EXISTENCE OF SOLUTIONS TO NONLINEAR ELLIPTIC EQUATIONS

by

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ABSTRACT

This dissertation is concerned with the existence of solutions to fully nonlinear elliptic equations of the form

$$\mathcal{A}u = \mathcal{F}u,$$

where $\mathcal{A}$ is a differential operator acting on a subspace of the Sobolev space $W^{1,p}_{loc}(\Omega)$, $p > 1$, Ω is a bounded domain in $\mathbb{R}^N$ and $\mathcal{F}$ is an operator depending on lower order terms which also satisfies certain growth conditions. In our study, we use variational methods, fixed point theorems and, especially, sub-supersolution theorems. Our sub-supersolution theorems obtained are motivated by and are more general than those of Vy Le and Schmitt. With our approach, the operator $\mathcal{F}$ is allowed to be singular, to contain convection terms and to involve nonlocal terms.

For my parents and my wife.

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NOTATIONS AND SYMBOLS

$\mathbb{R}^N$	N-dimensional Euclidean space.
p^*	$= \begin{cases} \frac{Np}{N-p} & \text{if } N > p \text{ and } N > 1, \\ \infty & \text{if } N \leq p \text{ or } N = 1, \end{cases}$ the Sobolev conjugate exponent of p.
Ω	A bounded open domain of $\mathbb{R}^N$ with smooth boundary.
$\partial\Omega$	The boundary of Ω.
∇u	$= (\partial u/\partial x_1, \partial u/\partial x_2, \ldots, \partial u/\partial x_n)$, the gradient of u.
$\text{div}(g)$	$= \partial g_1/\partial x_1 + \partial g_2/\partial x_2 + \ldots + \partial g_n/\partial x_n$, the divergence of g.
Δu	$= \text{div}(\nabla u)$, the Laplacian of u.
$\Delta_p u$	$= \text{div}(\|\nabla u\|^{p-2}\nabla u)$, the $p-$Laplacian of u.
$C^k(\Omega)$	The space of functions defined on Ω whose $k-th$ derivatives are continuous.
$C^k(\overline{\Omega})$	The restrictions of $C^k(\mathbb{R}^N)$ functions to $\overline{\Omega}$.
$C_0^k(\Omega)$	The space of $C^k(\Omega)$ functions whose supports are compact in Ω.
$C^{k,\alpha}(\Omega)$	The space of functions whose $k-th$ derivatives are $\alpha-$ Hölder continuous.
$C^\infty(\Omega)$	The space of functions whose derivatives of all order are continuous.
$C_0^\infty(\Omega)$	The space of $C^\infty(\Omega)$ functions with compact support.
$L^p(\Omega)$	The space of p integrable functions (the L^p norm is bounded).
$\\|f\\|_{W^{k,p}(\Omega)}$	$= \left(\sum_{\|\beta\| \leq k} \int_\Omega \|D^\beta f\|^p dx\right)^{1/p}$, the Sobolev norm.
$W^{1,p}(\Omega)$	The space of functions with bounded Sobolev norm.
$W_{loc}^{1,p}(\Omega)$	The space of functions with locally bounded Sobolev norm.
$W_0^{1,p}(\Omega)$	The closure of $C_0^\infty(\Omega)$ in $W^{k,p}(\Omega)$.
$\\|f\\|_X$	The norm of f in the normed space X.
$\\|f\\|$	$= \left(\int_\Omega \|f\|^p dx\right)^{\frac{1}{p}}$, the norm of f in $W_0^{1,p}(\Omega)$.
X^*	The dual space of a space X.

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CHAPTER 1

INTRODUCTION

The dissertation is concerned with sub-supersolution theorems and their applications to the study of boundary value problems involving the $p-$Laplace operator, $p > 1$, and some of its generalizations, where the $p-$Laplace operator is classically defined by

$$\Delta_p = \operatorname{div}(|\nabla|^{p-2}\nabla).$$

The $p-$Laplacian $-\Delta_p$ is also understood as a mapping from $W_0^{1,p}(\Omega)$ to $(W_0^{1,p}(\Omega))^*$ defined by

$$\langle -\Delta_p u, v\rangle = \int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v dx, \qquad \forall u, v \in W_0^{1,p}(\Omega),$$

when one seeks solutions in $W_0^{1,p}(\Omega)$ of partial differential equations in the weak sense. In particular, the $2-$Laplace operator is the usual Laplacian.

Generally, we study problems of the form

$$\mathcal{A}u = \mathcal{F}u \tag{1.1}$$

where $\mathcal{A}$ is a differential operator of order two acting on suitable function spaces and $\mathcal{F}$ is an operator depending on lower order terms which satisfies various growth conditions. The equations of this form appear in diverse areas such as shear bandings, fluid mechanics, non-Newtonian fluids, among others (see, e.g., [3, 4, 18]).

1.1 On positive solutions of elliptic equations

The first problem we are concerned with is the study of (1.1) when $\mathcal{F}$ is given by

$$\mathcal{F}u = \lambda f(u(\cdot)), \qquad u \in W_0^{1,p}(\Omega),$$

where λ is a parameter, f is a continuous function defined on $\mathbb{R}$ and $\mathcal{A} = -\Delta_p$. The results obtained here extend the papers [13, 27] in the sense that the Laplace operator in these two papers is replaced with its generalization, the $p-$Laplace operator, $p > 1$. The results can be applied to singular elliptic equations. More precisely, we study the following boundary value problem

$$\begin{cases} -\Delta_p u &= \lambda f(u) \quad \text{in } \Omega, \\ u &= 0 \quad\quad\quad \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

When $p = 2$, this problem becomes

$$\begin{cases} -\Delta u &= \lambda f(u) \quad \text{in } \Omega, \\ u &= 0 \quad\quad\quad \text{on } \partial\Omega \end{cases} \tag{1.3}$$

and neccessary and sufficient conditions for (1.3) to have multiple bounded solutions were studied by Peter Hess [27] and Dancer and Schmitt [13]. We recall here more details of their work.

In 1981, Peter Hess [27] established a multiple existence result for (1.3). His assumptions on f are:

1. $f(0) \geq 0$,

2. there exist $a_1 < b_1 < a_2 < b_2 < \cdots < b_{m-1} < a_m$, $m \geq 2$, such that for all $k = 1, \cdots, m-1$

$$\begin{cases} f(\cdot) \leq 0 & \text{on } (a_k, b_k), \\ f(\cdot) \geq 0 & \text{on } (b_k, a_{k+1}), \end{cases}$$

(see, e.g., Figure 1.1). Hess proved that if the function f satisfies

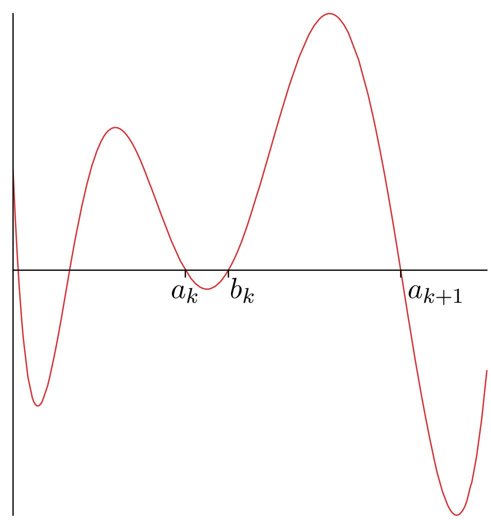

Figure 1.1. The graph of f.

$$\int_{a_k}^{a_{k+1}} f(s)ds > 0$$

for all $k \in \{1, \cdots, m-1\}$, then for all λ, sufficiently large, (1.3) has at least $m-1$ nonnegative solutions

$$\{u_1, \cdots, u_{m-1}\} \subset W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$$

such that

$$a_k < \|u_k\|_\infty \leq a_{k+1}$$

for each $k = 1, \cdots, m-1$. Later, Dancer and Schmitt [13] proved the converse of this result; namely, if (1.3) has a solution $u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ such that

$$a_k < \|u\|_\infty \leq a_{k+1}$$

for some $k \in \{1, \cdots, m-1\}$, then f must satisfy

$$\int_{a_k}^{a_{k+1}} f(s)ds > 0. \tag{1.4}$$

Motivated by these two results, we proved that when Δ is replaced by Δ_p, the conclusions of Hess [27] and Dancer and Schmitt [13] are still guaranteed. More precisely, the problem

$$\begin{cases} -\Delta_p u &= \lambda f(u) & \text{in } \Omega, \\ u &= 0 & \text{on } \partial\Omega, \end{cases}$$

has a solution u in $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ such that

$$a_k < \|u\|_\infty \leq a_{k+1},$$

for some $k \in \{1, 2, \cdots, m-1\}$, if, and only if,

$$\int_{a_k}^{a_{k+1}} f(s)ds > 0.$$

In order to obtain the results above, we establish a strong maximum principle which is similar to, but independent of, that of Vazquéz [55]. Indeed, we remove the requirement on the solution u that $\Delta u \in L^2_{loc}(\Omega)$ in Vazquéz' paper. Then, we use this maximum principle, variational methods and sub-supersolution theorems in [37, 38] to prove our main theorems. Note that the condition $f(0) \geq 0$ may be removed by using sub-supersolution theorems in [37, 38] again. Moreover, the results obtained can be used to study infinite semipositone elliptic problems; allowing that f be singular at 0, i.e, requiring that $f(0) = -\infty$.

1.2 Sub-supersolution theorems

The main purpose of this dissertation is to study partial differential equations and we consider sub-supersolution theorems as one of the main tools for us to do so. Versions of sub-supersolution theorems given here are motivated by the possible construction of well-order pairs of sub-supersolutions to a class of nonlinear singular boundary value problems, employing a first (or principal) positive eigenfunction of $-\Delta_p$.

We first extend some sub-supersolution theorems in [36, 37, 38] by allowing the presence of convection terms in the problems considered. More precisely, we establish serveral sub-supersolution theorems, which are applicable to problems of the form

$$\begin{cases} -\mathrm{div}A(x,\nabla u) &= f(x,u,\nabla u) \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{cases} \tag{1.5}$$

where the convection terms are understood as the dependence of f on ∇u and the mapping $A : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ is uniformly elliptic and satisfies Leray-Lions conditions. The presence of these convection terms causes the failure of variational methods because of the lack of a corresopding energy functional. In order to overcome this difficulty, we use cutting off techniques (introduced in [36, 37, 38]), approximating techniques, the Leray-Schauder degree function defined in the class (S_+) (see [6]), $W^{1,p}$ priori bounds (given in the Appendix) and some test functions (similar to those in [54]). Our approximating technique in this work is that we consider (1.5) in a sequence of subsomains of Ω. This motivates us to define the class $(\mathcal{S}_+)$. All details will be given in Chapter 3.

We then extend the results, introduced in the previous paragraph, by establishing sub-supersolution theorems for singular elliptic problems in two cases: with and without convection terms. Note that our theorems for the latter case are not more general than those for the former case although problems without convection terms can be considered as a particular case of those with convection terms (see Section 3.7 for the reason). The sub-supersolution theorem, established in this work, for singular problems without convection terms is a generalization of those in [36, 37, 38] in the sense that the growth condition is weakened and is proved by approximating the singular problems by nonsingular ones to which sub-supersolution theorems in [36, 37, 38] are applicable. This scheme is repeated when we prove sub-supersolution theorems for singular problems involving convection terms. Besides this approxomating technique, we shall again employ $W^{1,p}$ priori bounds on possible solutions, which will follow from certain growth conditions imposed on the nonlinear terms, the L^∞ bound of the solutions and some special test functions, similar to those in [54]. Our most general sub-supersolution theorem guarantees the existence of a minimal solution and a maximal solution (lying between the maximum of a finite number of given subsolutions and the minimum of some given supersolutions) to singular elliptic problems involving convection terms.

These sub-supersolution theorems are also presented in [45, 44].

1.3 Singular boundary problems without convection terms

We are concerned with singular elliptic equations, which arise in the study of non-Newtonian fluids, boundary layer phenomena for viscous fluids, chemical heterogenous catalysts, and in the theory of heat conduction in electrically conducting materials. Because of these many applications, such problems have been intensively studied (e.g., [9, 11, 12, 22, 33, 34, 52, 53, 56, 58]).

The following is the problem we are interested in:

$$\begin{cases} -\Delta_p u &=& ag(u)+\lambda h(u) & \text{in } \Omega, \\ u &>& 0 & \text{in } \Omega, \\ u &=& 0 & \text{on } \partial\Omega, \end{cases} \tag{1.6}$$

where λ is a nonnegative parameter;

$$a : \Omega \to [1, \infty)$$

is in $L^\infty(\Omega)$;

$$g : (0, \infty) \to [0, \infty)$$

is continuous and satisfies

$$\lim_{s\to 0^+} g(s) = \infty,$$

and

$$h : [0, \infty) \to \mathbb{R}$$

is continuous.

Two pioneering papers regarding such singular problems are those of Lazer and McKenna [34] and Crandall, Rabinowitz and Tartar [12]. These papers motivated a flurry of work in subsequent years (see, e.g., [9, 11, 22, 33, 52, 53, 56, 58]). All of these papers studied (1.6) in the case $p = 2$ and under the assumption that the singular term g either takes a particular form or satisfies a monotonicity condition. Thus the question arises whether or not the existence of solutions for (1.6) is still guaranteed when $p \in (1, \infty)$ and the monotonicity property is removed. Hai [24, 25] has given affirmative answers to this question in the case that Ω is an annulus, by establishing existence results for radial solutions which are solutions of associated ordinary differential equations.

As mentioned, we approach problem (1.6) by proving a version of a sub-supersolution theorem for singular elliptic problems and then finding such a well-ordered pair of sub-supersolutions for the specific singular problem under consideration. With this method, we can allow $p \in (1, \infty)$ and remove not only the monotonicity condition but also some technical conditions on the singular terms in the papers above.

Our main result can be summarized as follows.

Assume g satisfies:

$$\exists \gamma > 0,\ C > 0 \text{ such that } g(s) \leq C s^{-\gamma},\ \forall s \in (0, \infty). \tag{1.7}$$

Then:

(i) if $\limsup_{s \to 0^+} \dfrac{h(s)}{s^{p-1}} < \infty$, there exists $\tilde{\lambda} > 0$ such that for all $\lambda \in [0, \tilde{\lambda}]$, problem (1.6) has a solution,

(ii) if there exists $\alpha < p - 1$ such that

$$0 \leq h(s) \leq s^{\alpha},\ \forall s \in [1, \infty),$$

then for all $\lambda \geq 0$, problem (1.6) has a solution.

An interesting fact about the singular problem (1.6) is that when $\gamma \geq \frac{2p-1}{p-1}$, the solution obtained by the result above is not in $W_0^{1,p}(\Omega)$, which is usually true for $p-$Laplace equations. This is illustrated by the following example.

Example 1.1 in the case $N = 1$ and $\Omega = (0, 1)$, the function u defined by

$$x \longmapsto \sqrt{2x(1-x)}$$

does not belong to $W_0^{1,2}(0, 1)$ and is the unique solution of the boundary value problem

$$-u'' = u^{-3} \text{ in } \ (0,\, 1),\ u(0) = u(1) = 0.$$

1.4 Singular boundary value problems involving convection terms

The results presented here were motivated by [1, 21], studying singular boundary value problems for semilinear elliptic equations with convection terms. We cite the papers of Fulks and Maybe [18], Callegari and Nachman [7, 8] and some of their references, for giving physical situations from which such problems arise.

Consider the following problems

$$\begin{cases} -\Delta_p u &= g(x,u) + h(x,\nabla u) & \text{in } \Omega, \\ u &= 0 & \text{on } \partial\Omega, \end{cases} \tag{1.8}$$

where g and h are two Carathéodory functions defined on $\Omega\times(0,\infty)$ and $\Omega\times\mathbb{R}^N$, respectively, satisfying some growth conditions given in Subsection 4.2.1. These growth conditions allow g to be singular in the following sense:

$$\lim_{s\to 0} g(x,s) = \infty \text{ uniformly for } x \in \Omega.$$

Let $\phi > 0$ be a positive first eigenfunction of $-\Delta_p$. Then it is not hard to verify that $\epsilon\phi$ is a subsolution of (1.8) when $0 < \epsilon \ll 1$. On the other hand, the function $b\phi^t$ can be shown to be a supersolution of (1.8) by straight forward calculations and the choosing $b \gg 1$ and $t \in (0,1)$. Thus, the sub-supersolution theorems in Chapter 3 are applicable to solve this problem. This is one of our motivations to study sub-supersolution theorems. With this approach, we are able to obtain a more general result than that in [1] (see Theorem 4.12) in the sense that p must not necessarily equal to 2 and the Hölder continuity of h and g can be removed.

Next, motivated by [21], we study the following singular problems with the presence of a nonnegative parameter λ

$$\begin{cases} -\Delta_p u + k_1|\nabla u|^q &= k_2 g(u) + \lambda h(u) & \text{in } \Omega, \\ u &> 0 & \text{in } \Omega, \\ u &= 0 & \text{on } \partial\Omega, \end{cases} \tag{1.9}$$

where g is a continuous function and is singular at 0^+, and h is a continuous function defined on $\mathbb{R}$, which is called the parameter dependent term. Similar to the construction of the subsolution of (1.8), we can show that $\epsilon\phi$ is a subsolution of (1.9) when ϵ is sufficiently small. Moreover, a supersolution $\overline{u}$ of (1.9) is the solution of (1.6) obtained in Theorem 4.3. Note that Theorem 4.3 shows how the growth of h effects the existence of $\overline{u}$. Our results obtained here can be used to deduce Theorem 4.13, established in [21].

1.5 Nonlocal problems modeling shear bandings

One of the difficulties in studying problems (1.6), (1.8) and (1.9) is the fact that if $u \in W_0^{1,p}(\Omega)$ then, when x is close to $\partial\Omega$, $g(u(x))$ (or $g(x,u(x))$) blow up and, therefore, $g(u(\cdot))$ (or $g(\cdot,u(\cdot))$) might not belong to $(W_0^{1,p}(\Omega))^*$. More generally, we might understand that the problem

$$\begin{cases} -\Delta_p u &= f(x,u) & \text{in } \Omega, \\ u &= 0 & \text{on } \partial\Omega, \end{cases}$$

is singular if $f(\cdot, v(\cdot))$ might not be in $(W_0^{1,p}(\Omega))^*$ for some function $v \in W_0^{1,p}(\Omega)$. In particular, the Liouville-Gelfand-Bratu problem

$$\begin{cases} -\Delta_p u &= \lambda e^u \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{cases} \tag{1.10}$$

where λ is a nonnegative parameter, is singular (in dimension greater than 2). When Ω is a ball, this problem was considered in many papers (see, e.g., [5, 10, 29, 31, 42]) and the structure of radially symmetric solutions in the $\lambda - u$ plane was completely described in [31] when $p = 2$, in [29] for $p \in (1, \infty)$ and in [28] when Δ_p is replaced by the $k-$Hessian operators. Another singular crucial model for shear bandings, looking similar to but more general than (1.10), is the following nonlocal problem

$$\begin{cases} -\Delta_p u &= \frac{\lambda e^u}{\left(\int_\Omega e^u dy\right)^r} \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{cases} \tag{1.11}$$

where $r \geq 0$. It was first introduced and studied by Berbernes and Talaga [4] and then again by Miyasita [46] in 2007. They also described the structure of solutions in the $\lambda - u$ plane in the case Ω is a ball. A question arises whether or not problems (1.10) and (1.11) are solvable when Ω might not necessarily be a ball. We obtained results giving an affirmative answer. In particular, we are interested in solving the nonlocal problem

$$\begin{cases} -\Delta_p u &= \frac{\lambda f(x,u)}{\left(\int_\Omega h(y,u) dy\right)^r} \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{cases} \tag{1.12}$$

of which (1.10) and (1.11) are two particular cases, where $r \geq 0$ and Ω is smooth bounded domain of $\mathbb{R}^N$, which might not necessarily be a ball. In details, we use regularity results in [39, 40] and the Ascoli-Azelà theorem to show that the map

$$\begin{aligned} L_\lambda : C^1(\overline{\Omega}) &\to C^1(\overline{\Omega}) \\ u &\mapsto (-\Delta_p)^{-1} \left(\frac{\lambda f(x,u)}{(\int_\Omega h(y,u) dy)^r} \right) \end{aligned}$$

is completely continuous. Then employing Leray-Schauder continuation arguments (see [16, 49]), we conclude that problem

$$u - L_\lambda u = 0,$$

has an unbounded continuum of solutions in $\mathcal{C} \subset [0, \infty) \times C^1(\overline{\Omega})$.

This result implies that for λ small, (1.10) and (1.11) have solutions. However, we do not know how far λ can be away from 0 such that the existence result is still guaranteed. In order to study this, we note that if (1.10) has a solution, then such a solution is a supersolution of (1.11). This and the fact that 0 is a subsolution of (1.11) suggest to us to use cutting off technique and sub-supersolution methods. The details will be given in Section 5.2.

CHAPTER 2

ON POSITIVE SOLUTIONS OF ELLIPTIC EQUATIONS

In 1981, Peter Hess [27] established a multiplicity result for solutions of boundary value problems for nonlinear perturbations of the Laplace operator. The sufficient conditions given were later shown to be also necessary by Dancer and Schmitt [13]. In this chapter, we show that similar (and slightly more general) results hold when the Laplace operator is replaced by the $p-$Laplacian. Some applications to singular problems are given, as well.

2.1 Main statements

Let f be a continuous function on $\mathbb{R}$ and assume throughout that f satisfies:

(i) $f(0) \geq 0$,

(ii) there exist

$$0 < a_1 < b_1 < a_2 < b_2 < \cdots < b_{m-1} < a_m,$$

such that for all $k = 1, \cdots, m-1$,

$$\begin{cases} f(\cdot) \leq 0 & \text{on } (a_k, b_k), \\ f(\cdot) \geq 0 & \text{on } (b_k, a_{k+1}). \end{cases}$$

Motivated by the results in [13] (for the case $p = 2$), we establish sufficient conditions in order that the problem

$$\begin{cases} -\Delta_p u &= \lambda f(u) \quad \text{in } \Omega, \\ u &= 0 \qquad \text{on } \partial\Omega \end{cases} \tag{2.1}$$

has, for $\lambda >> 1$, at least $m-1$ nonnegative weak solutions

$$\{u_1, \cdots, u_{m-1}\} \subset W_0^{1,p}(\Omega) \cap L^\infty(\Omega),$$

such that

$$a_k < \|u_k\|_\infty \leq a_{k+1}, \; k = 1, \cdots, m-1.$$

The following are the results to be established in the chapter:

Theorem 2.1 *Assume that the function f satisfies*

$$\int_{a_k}^{a_{k+1}} f(s)ds > 0 \text{ for all } k \in \{1, \cdots, m-1\}, \tag{2.2}$$

then for all λ, sufficiently large, problem (2.1) has at least $m-1$ nonnegative solutions $u_1, \cdots, u_{m-1} \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ such that $a_k < \|u_k\|_\infty \leq a_{k+1}$ for each $k = 1, \cdots, m-1$.

We further have necessary conditions given by the following result.

Theorem 2.2 *Assume that the problem*

$$\begin{cases} -\Delta_p u &= f(u) \quad \text{in } \Omega, \\ u &= 0 \qquad \text{on } \partial\Omega \end{cases} \tag{2.3}$$

has a nonnegative weak solution u such that $\|u\|_\infty \in (a_k, a_{k+1}]$, then

$$\int_{a_k}^{a_{k+1}} f(s)ds > 0,$$

for $k \in \{1, \cdots, m-1\}$.

The proofs of these theorems follow the ideas used in and extensions of the proofs in the papers [27] and [13], which have been suitably modified and expanded for the case being considered. We shall also provide some remarks and discuss applications to problems whose nonlinear terms may be become singular at 0.

2.2 Proof of Theorem 2.1

In this section, we give a proof of Theorem 2.1. The proof follows Hess' [27] arguments very closely. We first need a lemma which is a consequence of the weak maximum principle for the $p-$Laplace operator.

Lemma 2.3 *Let $g : \mathbb{R} \to \mathbb{R}$ be a continuous function such that there exists $s_0 \geq 0$ such that $g(s) \geq 0$, if $s \in (-\infty, 0)$ and $g(s) \leq 0$, if $s \geq s_0$. If u is a weak solution of*

$$-\triangle_p u = g(u) \text{ in } \Omega, \ u = 0 \text{ on } \partial\Omega, \tag{2.4}$$

then u is nonnegative a.e. and belongs to $L^\infty(\Omega)$. Moreover,

$$\|u\|_\infty \leq s_0.$$

Proof : We let $v = u^- = \max\{-u, 0\}$, then $v \in W_0^{1,p}(\Omega)$ and

$$\nabla v = \begin{cases} -\nabla u & u < 0 \\ 0 & u \geq 0. \end{cases}$$

Hence, since u is a weak solution of (2.4), we have

$$\int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla v dx = \int_\Omega g(u) v dx.$$

This implies $\|v\| \leq 0$ and therefore $v = 0$, and $u \geq 0$ a.e. on Ω. Next, choosing the test function $v = (u - s_0)^+ = \max\{u - s_0, 0\} \in W_0^{1,p}(\Omega)$ in the equation

$$\int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla v dx = \int_\Omega g(u) v dx,$$

we have $\|v\| \leq 0$ and therefore $u \leq s_0$ a.e., i.e., $\|u\|_\infty \leq s_0$.

For $k = 2, \cdots, m$, define f_k as follows:

$$f_k(s) := \begin{cases} f(0) & s \leq 0 \\ f(s) & 0 \leq s \leq a_k \\ 0 & s > a_k, \end{cases}$$

and let

$$F_k(s) = \int_0^s f_k(\sigma) d\sigma.$$

For any $\lambda \geq 0$, let the functional

$$\Phi_k(\lambda, \cdot) : W_0^{1,p}(\Omega) \to \mathbb{R}$$

be defined by

$$\Phi_k(\lambda, u) = \frac{1}{p} \int_\Omega |\nabla u|^p - \lambda \int_\Omega F_k(u) dx$$

and denote by $K_k(\lambda)$ the set of critical points of $\Phi_k(\lambda, \cdot)$. Then, if u is in $K_k(\lambda)$, u is a weak solution of

$$\begin{cases} -\triangle_p u &= \lambda f_k(u) \quad \text{in } \Omega, \\ u &= 0 \quad\quad\quad \text{on } \partial\Omega. \end{cases}$$

By Lemma 2.3, u is nonnegative and $\|u\|_\infty \leq a_k$. We have thus shown:

Lemma 2.4 *u is in $K_k(\lambda)$, if, and only if, u is a nonnegative weak solution of (2.1) and belongs to $L^\infty(\Omega)$ with $\|u\|_\infty \leq a_k$.*

We next claim that $K_k(\lambda)$ is not empty. Since f_k is bounded and vanishes on (a_k, ∞), $\Phi_k(\lambda, \cdot)$ is coercive. Futher, since the first summand defining $\Phi_k(\lambda, \cdot)$ is $\frac{1}{p}\|\cdot\|^p$, it is continuous and since it is a convex functional, it is weakly lower semicontinuous. The second summand is weakly continuous, as follows from the compact embedding of $W_0^{1,p}(\Omega)$ in $L^p(\Omega)$. Thus, there exists $u_k(\lambda)$ such that

$$\Phi_k(\lambda, u_k(\lambda)) = \inf\{\Phi_k(\lambda, v) : v \in W_0^{1,p}(\Omega)\}.$$

The following lemma shows that for $k = 2, \cdots, m$, $a_{k-1} < \|u_k\|_\infty \leq a_k$ and therefore, (2.1) has at least $m-1$ solutions when $\lambda > 0$ sufficiently large.

Lemma 2.5 *For each $k = 2, \cdots, m$, there exists $\lambda_k > 0$, such that for all $\lambda > \lambda_k$, $u_k(\lambda) \notin K_{k-1}(\lambda)$.*

Proof : We show that there exist $\lambda_k > 0$ and $w \in W_0^{1,p}(\Omega)$, $w \geq 0$, and $\|w\|_\infty \leq a_k$, such that

$$\Phi_k(\lambda, w) < \Phi_{k-1}(\lambda, u_{k-1}(\lambda)), \qquad \forall \lambda > \lambda_k.$$

This will imply the assertion.

Since f satisfies (2.2),

$$0 < \alpha := F(a_k) - \max\{F(s) : 0 \leq s < a_{k-1}\},$$

where $F(s) = \int_0^s f(\sigma) d\sigma$. Then for all u in $W_0^{1,p}(\Omega)$ satisfying $0 \leq u \leq a_{k-1}$ a.e.,

$$\int_\Omega F(a_k) dx \geq \int_\Omega F(u) dx + \alpha|\Omega|,$$

where $|\Omega|$ is the Lebesgue measure of Ω.

For $\delta > 0$, let $\Omega_\delta := \{x \in \Omega : \mathrm{dist}(x, \partial\Omega) < \delta\}$. By Lebesgue's dominated convergence theorem, the measure $|\Omega_\delta| \to 0$ as $\delta \to 0$. On the other hand, for each $\delta > 0$, there exists $w_\delta \in C_0^\infty(\Omega)$ with $0 \leq w_\delta \leq a_k$, $w_\delta(x) = a_k$ $\forall x \in \Omega \setminus \Omega_\delta$. Thus

$$\begin{aligned}
\int_\Omega F(w_\delta) dx &= \int_{\Omega \setminus \Omega_\delta} F(a_k) dx - \int_{\Omega_\delta} F(w_\delta) dx \\
&= \int_\Omega F(a_k) dx - \int_{\Omega_\delta} (F(a_k) + F(w_\delta)) dx \\
&\geq \int_\Omega F(a_k) dx - 2C|\Omega_\delta|,
\end{aligned}$$

where $C = \max\{|F(s)| : 0 \leq s \leq a_k\}$. Hence, for all u in $W_0^{1,p}(\Omega)$ with $0 \leq u \leq a_{k-1}$,

$$\int_\Omega F(w_\delta) dx \geq \int_\Omega F(u) dx + \alpha|\Omega| - 2C|\Omega_\delta|.$$

Fix $\delta > 0$ such that $\eta := \alpha|\Omega| - 2C|\Omega_\delta| > 0$ and set $w := w_\delta$. Then for all u, $0 \leq u \leq a_{k-1}$,

$$\begin{aligned}
\Phi_k(\lambda, w) - \Phi_{k-1}(\lambda, u) &= \frac{1}{p}(\|w\|^p - \|u\|^p) - \lambda \int_\Omega (F(w) - F(u)) dx \\
&\leq \frac{1}{p}\|w\|^p - \lambda\eta < 0,
\end{aligned}$$

provided $\lambda > 0$ is chosen sufficiently large.

Thus, for all λ large enough, there are $m-1$ solutions

$$u_1(\lambda), \cdots, u_{m-1}(\lambda)$$

as asserted.

2.3 Proof of Theorem 2.2

It follows easily that it will suffice to consider the case that $k = 1$. We also assume here that $f(0) > 0$ and remove this condition later. Throughout this section, all bounded weak solutions u of (2.1) are in class $C^1(\Omega)$ because of Theorem 1.7 in [40]. We first establish a strong maximum principle for weak solutions of equation (2.1).

Lemma 2.6 *Let $u \in C^1(\Omega)$ be a nonnegative weak solution of (2.3). If $f(0) > 0$ then u is positive in Ω.*

Proof : Assume there exists x_0 in Ω such that $u(x_0) = 0$. Let D be a ball contained in Ω such that $x_0 \in \partial D$. Denote by y_0 and r the center and the radius of D, respectively, and let $g \leq f$ be a strictly decreasing continuous function defined on $[0, \infty)$ such that $g(0) = f(0) > 0$ and $\gamma := g(\frac{a_1}{2}) = \inf\{g(s) : 0 \leq s \leq \frac{a_1}{2}\} > 0$. Now, define a function b on D as

$$b(x) = \epsilon \left(e^{-|\frac{x-y_0}{r}|^2} - e^{-1} \right),$$

where ϵ is sufficiently small such that

$$\sup_{x \in D} |\mathrm{div}(|\nabla b(x)|^{p-2} \nabla b(x))| \leq \gamma.$$

Then b is a subsolution of

$$\begin{cases} -\mathrm{div}(|\nabla u|^{p-2} \nabla u) = g(u) & \text{in } D, \\ u = 0 & \text{on } \partial D. \end{cases}$$

It follows that for all $\varphi \geq 0$ in $W_0^{1,p}(D)$,

$$\int_D (|\nabla b|^{p-2} \nabla b - |\nabla u|^{p-2} \nabla u) \cdot \nabla \varphi dx \leq \int_D (g(b) - g(u)) \varphi dx.$$

Choosing $\varphi = (b-u)^+$, and using the fact that the $p-$Laplace operator is monotone and g is strictly decreasing, we obtain

$$\begin{aligned} 0 &\leq \int_{D^+} (|\nabla b|^{p-2} \nabla b - |\nabla u|^{p-2} \nabla u) \cdot \nabla (b-u) dx \\ &\leq \int_{D^+} (g(b) - g(u))(b-u) dx \leq 0, \end{aligned}$$

where $D^+ = \{x \in D : b(x) > u(x)\}$. Therefore, D^+ is empty or equivalently $u \geq b$ in D. Since $u(x_0) = b(x_0) = 0$, and $b > 0$ in D we have that the normal derivative with respect to the boundary of D satisfies $\partial_\nu u(x_0) \leq \partial_\nu b(x_0) < 0$, implying that $|\nabla u(x_0)| \neq 0$, which contradicts that $u(x_0) = 0$ is a minimum value of u in Ω.

Let B be an open ball centered at 0 and containing Ω. Define the function α on Ω by

$$\alpha(x) = \begin{cases} u(x) & x \in \bar{\Omega} \\ 0 & x \in \bar{B} \setminus \Omega. \end{cases}$$

Since Ω is a domain with smooth boundary, $\alpha \in W_0^{1,p}(B)$. We also have:

Lemma 2.7 *α is a subsolution of*

$$\begin{cases} -\Delta_p u &= f(u) & \text{in } B, \\ u &= 0 & \text{on } \partial B. \end{cases} \tag{2.5}$$

Proof : For each n in $\mathbb{N}$, define $v_n(x) = n \min\{u(x), \frac{1}{n}\}$, $x \in \Omega$, where u is as in Lemma 2.6. Then $\nabla u \cdot \nabla v_n$ is nonnegative and by Lemma 2.6, $\{v_n\}$ converges to 1 pointwise in Ω. Let $w \geq 0$ be in $C_0^\infty(B)$. Then wv_n is in $W_0^{1,p}(\Omega)$ and since u is a weak solution of (2.1),

$$\int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla (wv_n) dx = \int_\Omega f(u) wv_n dx,$$

or

$$\int_\Omega w|\nabla u|^{p-2} \nabla u \cdot \nabla v_n dx + \int_\Omega v_n |\nabla u|^{p-2} \nabla u \cdot \nabla w dx = \int_\Omega f(u) wv_n dx.$$

Now, applying Lebesgue's dominated convergence theorem and noting that $0 \leq v_n \leq 1$ for all n, we have

$$\begin{aligned} \int_B |\nabla \alpha|^{p-2} \nabla \alpha \cdot \nabla w dx &= \int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla w dx \\ &= \lim_{n\to\infty} \int_\Omega v_n |\nabla u|^{p-2} \nabla u \cdot \nabla w dx \\ &= \lim_{n\to\infty} \int_\Omega \left(f(u) wv_n - w|\nabla u|^{p-2} \nabla u \cdot \nabla v_n \right) dx \\ &\leq \int_\Omega f(u) w dx \\ &\leq \int_B f(\alpha) w dx, \end{aligned}$$

proving the lemma.

The following demonstrates Theorem 2.2, which is an extention of a result of [13].

Theorem 2.8 *Assume that $f(0)$ is positive. If (2.1) has a nonnegative weak solution u in $L^\infty(\Omega)$ such that $\|u\|_\infty \in (a_1, a_2]$, then*

$$\int_{a_1}^{a_2} f(s) ds > 0.$$

We remark again that the condition $f(0) > 0$ will be removed later.

Proof : Define $\beta(x) = a_2$ for all x in B. Since β and α are a supersolution and a subsolution, respectively, (2.5) has a maximum solution $\bar{u}$ such that $\alpha(x) \le \bar{u}(x) \le a_2$, for all x in B (see Remark 1.5 in [38]). This means, for all solution v of (2.5) with $\alpha(x) \le v(x) \le a_2$, $v \le \bar{u}$. We claim that this implies that $\bar{u}$ is radially symmetric, i.e. $\bar{u}(x_1) = \bar{u}(x_2)$ for all x_1, x_2 in B such that $|x_1| = |x_2|$. Assuming this is not the case, there exist x_1 and x_2 in B with $|x_1| = |x_2|$ such that $\bar{u}(x_1) < \bar{u}(x_2)$. Let P be a $N \times N$ matrix in $SO(N, \mathbb{R})$, the special orthogonal group, such that $x_2 = Px_1$. Note that the transpose matrix P^T of P is also its inverse matrix. Let $u_1(x) = \bar{u}(Px)$. Since for all x in Ω

$$\nabla u_1(x) = P\nabla \bar{u}(Px),$$

and the map $x \mapsto Px$ is an isometry, it follows that

$$|\nabla u_1(x)| = |P\nabla \bar{u}(Px)| = |\nabla \bar{u}(Px)|.$$

We next show that u_1 is a weak solution of (2.5). That is, we need to verify for all φ in $W_0^{1,p}(B)$

$$\int_B |\nabla u_1(x)|^{p-2}\nabla u_1(x) \cdot \nabla\varphi(x)dx = \int_B f(u_1(x))\varphi(x)dx. \tag{2.6}$$

Let $\psi(x) = \varphi(P^T x) \in W_0^{1,p}(B)$. The left-hand side of (2.6) becomes

$$\begin{aligned}
&\int_B |\nabla \bar{u}(Px)|^{p-2} P\nabla \bar{u}(Px) \cdot \nabla\varphi(x)dx \\
&\qquad = \int_B |\nabla \bar{u}(Px)|^{p-2} P\nabla \bar{u}(Px) \cdot (P\nabla\psi(Px))\, dx \\
&\qquad = \int_B |\nabla \bar{u}(y)|^{p-2} \nabla \bar{u}(y) \cdot \nabla\psi(y) \det P dy \\
&\qquad = \int_B f(\bar{u}(y))\psi(y)dy \\
&\qquad = \int_B f(\bar{u}(PP^T y))\psi(PP^T y)dy \\
&\qquad = \int_B f(\bar{u}(Px))\psi(Px) \det P^T dx \\
&\qquad = \int_B f(u_1(x))\varphi(x)dx.
\end{aligned}$$

Hence (2.6) holds.

Now, (2.5) has two subsolutions, α and u_1. It follows from Theorem 1.4 in [38] that (2.5) has another solution u_2 such that

$$\max\{\alpha, u_1\} \le u_2 \le \beta.$$

Since $\bar{u}$ is the maximum solution with respect to the pair of sub-supersolutions (α, β), we have

$$\bar{u}(x_1) \ge u_2(x_1) \ge u_1(x_1) = \bar{u}(x_2) > \bar{u}(x_1).$$

This contradiction shows that $\bar{u}$ is radially symmetric.

Next, define a C^1-function $u : [0, R) \to \mathbb{R}^+$ by $u(|x|) = \bar{u}(x)$ for all x in B, where R is the radius of B. Using the chain rule for classical differentiation, we have for all r in $(0, R)$

$$\begin{aligned} \frac{\partial \bar{u}}{\partial x_i} &= \frac{du}{dr}\frac{\partial r}{\partial x_i} = u'\frac{x_i}{r}, \ i = 1, ..., N \\ |\nabla \bar{u}| &= |u'| \left(\sum_{i=1}^{N} \frac{x_i^2}{r^2} \right)^{\frac{1}{2}} = |u'|. \end{aligned}$$

For any v in $C_0^\infty(0, R)$, put

$$w(r) = \frac{v(r)}{r^{N-1}}, \qquad r \in (0, R), \ w(0) = 0$$

and

$$\bar{v}(x) = v(|x|), \ \bar{w}(x) = w(|x|), \qquad x \in B.$$

Now, as a weak solution of (2.5), $\bar{u}$ satisfies

$$\int_B |\nabla \bar{u}|^{p-2} \nabla \bar{u} \cdot \nabla \bar{w} dx = \int_B f(\bar{u}) \bar{w} dx.$$

But, $\frac{\partial \bar{w}(x)}{\partial x_i} = w' \frac{x_i}{|x|}$, and thus,

$$\int_0^R |u'|^{p-2} u' w' r^{N-1} dr = \int_0^R f(u) w r^{N-1} dr.$$

Substituting $w = \frac{v}{r^{N-1}}$ and $w' = \frac{v'}{r^{N-1}} - \frac{N-1}{r^N} v$ into this equation, we obtain

$$\int_0^R |u'|^{p-2} u' \left(\frac{v'}{r^{N-1}} - \frac{N-1}{r^N} v \right) r^{N-1} dr = \int_0^R f(u) \frac{v}{r^{N-1}} r^{N-1} dr,$$

or

$$\int_0^R |u'|^{p-2} u' v' dr - \int_0^R \frac{N-1}{r} |u'|^{p-2} u' v dr = \int_0^R f(u) v dr,$$

for all v in $C_0^\infty(0, R)$. This implies that u is a C^1 weak solution of the equation

$$-\partial(|u'|^{p-2}u') = \frac{N-1}{r}|u'|^{p-2}u' + f(u), \tag{2.7}$$

and by the continuity of the right-hand side, the distributional derivative ∂ above becomes a classical derivative and hence u is a classical solution of (2.7).

Since $\bar{u}$ is radially symmetric, $u'(0) = 0$. Hence, u is a solution of (2.7) subject to the condition $u'(0) = 0 = u(R)$. Let $r_0 \in [0, R)$ such that $u_{\max} = u(r_0) = \max\{u(r) : r \in [0, R)\}$. Multiplying both sides of (2.7) by u' and intergrating it, we obtain

$$-\left(\int_{r_0}^{r} (p-1)|u'|^{p-2}u'u''d\tau + (N-1)\int_{r_0}^{r} \frac{|u'|^p}{\tau}d\tau\right) = \int_{r_0}^{r} f(u)u'd\tau,$$

for all $0 < r < R$. Now, since $u_{\max} = u(r_0)$ is greater than a_1, we can choose $r \in (0, R)$ such that $u(r) = a_1$. The above equality becomes

$$\int_{u_{\max}}^{a_1} f(s)ds = -(p-1)\int_{0}^{u'(r)} |s|^{p-2}sds - (N-1)\int_{r_0}^{r} \frac{|u'|^p}{\tau}d\tau < 0.$$

This equation shows $\int_{a_1}^{u_{\max}} f(s)ds > 0$. Because $f \leq 0$ in $(a_1, b_1]$, $u_{\max} \in (b_1, a_2]$ and f is nonnegative in $[u_{\max}, a_2]$, we get the desired result because

$$\int_{a_1}^{a_2} f(s)ds \geq \int_{s_0}^{u_{\max}} f(s)ds > 0.$$

2.4 Some remarks

Remark 2.9 We can remove the condition $f(0) > 0$ in Theorem 2.8.

Proof : Assume that $f(0) \leq 0$ and again assume that (2.1) has a nonnegative solution u in $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ satisfying $\|u\|_\infty \in (a_1, a_2]$. Let $\tilde{f}$ be a continuous function so that $\tilde{f}(0) > 0$, $\tilde{f}(s) \geq f(s)$ when $0 \leq s \leq a_1$ and $\tilde{f}(s) = f(s)$ on $[a_1, \infty)$. Then u is a subsolution of

$$\begin{cases} -\triangle_p u = \tilde{f}(u) & x \in \Omega, \\ u = 0 & x \in \partial\Omega, \end{cases} \tag{2.8}$$

and as before, we may use $\beta(x) \equiv a_2$ as a supersolution for (2.8). Hence, (2.8) has a solution $\tilde{u}$ satisfying $u \leq \tilde{u} \leq a_2$. We now proceed as in the first part of the proof with $\tilde{f}$ in place of f and obtain

$$\int_{a_1}^{a_2} f(s)ds = \int_{a_1}^{a_2} \tilde{f}(s)ds > 0.$$

The previous remark and Theorem 2.8 consider the case $0 < a_1 < b_1 < a_2$. Now, we study the case $0 = a_1 < b_1 < a_2$.

Remark 2.10 Let $a < b$ be two positive numbers and let f be a continuous function on $[0, b]$ such that $f(0) = 0$, $f < 0$ on $(0, a)$, $f > 0$ on (a, b) and $f(a) = f(b) = 0$. Assume the problem

$$\begin{cases} -\triangle_p u = f(u) & x \in \Omega, \\ u = 0 & x \in \partial\Omega \end{cases} \tag{2.9}$$

has a nonnegative weak solution u in $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ such that $\|u\|_\infty$ is in $(a, b]$ then

$$\int_0^b f(s)ds \geq 0.$$

Proof : Let $\epsilon < a$ be an arbitrarily small positive number. For each n in $\mathbb{N}$, define a continuous function g_n satisfying $g_n(0) = 1$, $g_n > 0$ on $(0, \epsilon)$, $f \leq g_n < 0$ on (ϵ, a), $g_n = f$ on $[a, b]$ and $\int_\epsilon^a g_n(s)ds < \int_0^a f(s)ds + \frac{1}{n}$ (see, e.g., Figure 2.1). Since u is a solution of (2.9) and $g_n \geq f$, u is a subsolution of

$$\begin{cases} -\triangle_p u &= g_n(u) \quad x \in \Omega, \\ u &= 0 \qquad\quad x \in \partial\Omega. \end{cases} \tag{2.10}$$

Now (2.10) has a pair of sub-supersolutions (u, b), and, as in the proof of Theorem 2.8, (2.10) has a weak solution u_1 such that $u \leq u_1 \leq b$ a.e. in Ω. Using Theorem 2.8 and the assumption on g_n, we have

$$\int_0^b f(s)ds + \frac{1}{n} > \int_\epsilon^b g_n(s)ds > 0,$$

from which the assertion follows.

Let us next consider the problem (2.3) under the assumption that $f(0)$ is $-\infty$. Such problems have been studied extensively in recent years, see, e.g., [19, 47, 57]. To give a particular example, we show that one such result, Theorem 1.1 in [19], may be deduced from Theorem 2.8 and Remark 2 in the case all functions in the problem in [19] are independent of x. Of course, our result is somewhat more general than Theorem 1.1 in [19] since we are

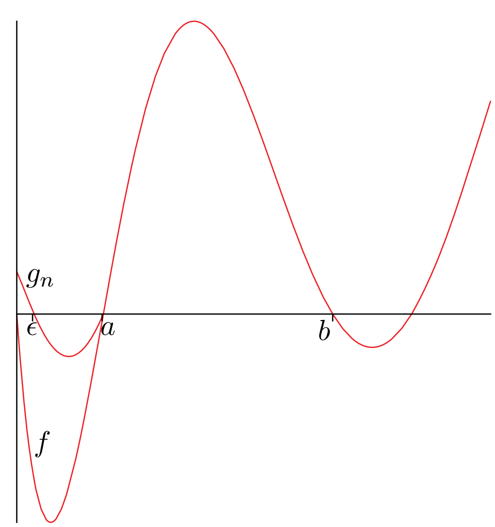

Figure 2.1. Graphs of f and g_n.

considering a problem of $p-$Laplacian type and also remove some smoothness and growth conditions required in that paper. This idea will be given in Remark 2.11.

Let g be continuous on $(0,\infty)$ and $\lim_{s\to 0^+} g(s)=\infty$. Let $h:[0,\infty)\to[0,\infty)$ be continuous. Assume there exists $s_0>0$ such that $f=-g+h$ is negative on $(0,s_0)$ and positive on (s_0,∞). The following holds:

Remark 2.11 If $\int_0^{s_0} g(s)ds=\infty$ then the problem

$$\begin{cases} -\Delta_p u+g(u) &= h(u) \quad \text{in } \Omega,\\ u &> 0 \quad \text{in } \Omega,\\ u &= 0 \quad \text{on } \partial\Omega \end{cases} \tag{2.11}$$

has no nonnegative weak solution in $L^\infty(\Omega)$.

Proof : Assume that u is a nonnegative weak solution of (2.11) such that

$$M=\|u\|_\infty<\infty.$$

It follows that $M>s_0$. For, if $M\le s_0$, then

$$0\le\int_\Omega|\nabla u|^p dx=\int_\Omega(-g(u)+h(u))u dx\le 0$$

and thus, $u=0$.

Define a continuous function f on $(0,\infty)$ and a sequence of continuous functions $\{f_n\}_{n\in\mathbb{N}}$ on $[0,\infty)$ such that

$$f(s)=\begin{cases} -g(s)+h(s) & s\in(0,M),\\ \ge 0 & s\in(M,M+1),\\ 0 & s>M+1 \end{cases}$$

and

$$\begin{cases} f_n(0)=0, & \\ f(s)\le f_n(s)\le 0 & s\in(0,\frac{1}{n}],\\ f_n(s)=f(s) & s\in(\frac{1}{n},\infty). \end{cases}$$

(see, e.g., Figure 2.2)

Since $u\le M$ a.e. in Ω and $f_n\ge f$ for all $n\in\mathbb{N}$, u is a subsolution of

$$\begin{cases} -\Delta_p u &= f_n(u) \quad \text{in } \Omega,\\ u &= 0 \quad \text{on } \partial\Omega \end{cases} \tag{2.12}$$

for all $n\in\mathbb{N}$. Applying Theorem 1.4 in [38], we deduce that (2.12) has a weak solution u_n such that $u\le u_n\le M+1$ a.e. in Ω. Now, Remark 2 implies that

$$\int_0^{M+1} f_n(s)ds\ge 0$$

and

$$\int_0^{s_0} -f_n(s)ds\le\int_{s_0}^{M+1} f_n(s)ds=\int_{s_0}^{M+1} f(s)ds$$

for all $n\in\mathbb{N}$.

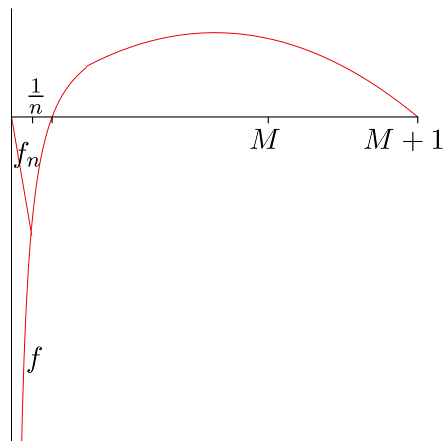

Figure 2.2. Graphs of f and f_n.

On the other hand, $-f_n(s)$ converges to $-f(s) = g(s) - h(s)$ for every s in $(0, s_0)$ as n approaches ∞. Noting that $-f_n \geq 0$ on $(0, s_0)$, we can use Fatou's Lemma to get

$$\int_0^{s_0} (g(s) - h(s))ds \leq \liminf_{n\to\infty} \int_0^{s_0} -f_n(s)ds \leq C < \infty,$$

where $C = \int_{s_0}^{M+1} f(s)ds$ is a constant. It follows that $\int_0^{s_0} g(s)ds < \infty$ which contradicts the hypothesis.

CHAPTER 3

SUB-SUPERSOLUTION THEOREMS

The use of sub-supersolution theorems is a powerful tool to study partial differential equations. Their general statement is that under some conditions, if an equation has a well-ordered pair of sub-supersolutions, then it has a solution lying between such a pair. More generally, some versions of sub-supersolution theorems, showing the existence of minimal and maximal solutions to an elliptic equation staying between the maximum of several subsolutions and the minimum of some supersolutions, can be found in [36, 37, 38]. Since these theorems play important roles in the proofs of our sub-supersolution theorems, we shall recall them in the Appendix. Unfortunately, some growth conditions imposed in these theorems do not hold if the equation considered contains a singular term. This motivates us to establish some versions of sub-supersolution theorems which are applicable to singular equations. We are also motivated by [1, 15, 20, 21] to prove sub-supersolution theorems for singular equations involving convections terms.

3.1 General setting for the principal operator

For any open set $\mathcal{O} \subset \mathbb{R}^m$, $m \in \{1, 2, \cdots\}$, a mapping $H : \Omega \times \mathcal{O} \to \mathbb{R}$ is said to be a Carathéodory function if, and only if:

(i) $H(\cdot, s)$ is measurable for all $s \in \mathcal{O}$,

(ii) $H(x, \cdot)$ is continuous for a.e. $x \in \Omega$.

We consider the following problem

$$\mathcal{A}u = \mathcal{F}u. \tag{3.1}$$

Here, $\mathcal{A}$ is defined by the Carathéodory mapping

$$A : \Omega \times \mathbb{R}^N \to \mathbb{R}^N,$$

with

$$\langle \mathcal{A}u, v \rangle = \int_\Omega A(x, \nabla u) \cdot \nabla v dx, \tag{3.2}$$

and $\mathcal{F}$ is defined by the Carathéodory function f whose domain depends on the type of the problems studied with

$$\langle \mathcal{F}u, v\rangle = \int_\Omega fv dx \tag{3.3}$$

for all u, v in some suitable functional spaces given later.

We begin with some assumptions of A. Suppose that A is a Carathéodory function and satisfies:

$$|A(x,\xi)| \leq a_1(x) + b_1|\xi|^{p-1}, \qquad \text{for a.e. } x \in \Omega, \text{ all } \xi \in \mathbb{R}^N, \tag{3.4}$$

with $p \in (1,\infty)$ (fixed), $a_1 \in L^{\frac{p}{p-1}}(\Omega)$ and $b_1 > 0$. Moreover, A is assumed to be strictly monotone; i.e.,

$$(A(x,\xi) - A(x,\xi')) \cdot (\xi - \xi') > 0, \tag{3.5}$$

for a.e. $x \in \Omega$, all $\xi, \xi' \in \mathbb{R}^N$, $\xi \neq \xi'$, and A is coercive in the following sense: there exist $a_2 \in L^1(\Omega)$ and $b_2 > 0$ such that

$$A(x,\xi) \cdot \xi \geq b_2|\xi|^p - a_2(x), \qquad \text{for a.e. } x \in \Omega, \text{ all } \xi \in \mathbb{R}^N. \tag{3.6}$$

Assume furthermore that the map $\mathcal{A} : W_0^{1,p}(\Omega) \to (W_0^{1,p}(\Omega))^*$ is of class $(\mathcal{S}_+)$, where $(\mathcal{S}_+)$ is defined as follows.

Definition 3.1 *Let $L : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ be such that the map $\mathcal{L}$ defined by*

$$\langle \mathcal{L}u, v\rangle := \int_\Omega L(x, \nabla u) \cdot \nabla v dx$$

satisfies

$$\mathcal{L} : W_0^{1,p}(\Omega) \to (W_0^{1,p}(\Omega))^*.$$

We say that $\mathcal{L}$ belongs to class $(\mathcal{S}_+)$ if, and only if: for all sequences $\{u_n\}_{n\in\mathbb{N}}$ converging weakly to u in $W_0^{1,p}(\Omega)$, whenever $\Omega' \subset \Omega$ is such that

$$\limsup_{n\to\infty} \int_{\Omega'} L(x, \nabla u_n) \cdot \nabla(u_n - u) dx \leq 0,$$

then

$$\nabla u_n \to \nabla u \quad \text{ in } (L^p(\Omega'))^N.$$

For any sequence $\{u_n\}_{n\in\mathbb{N}}$ converging weakly to a function u in $W_0^{1,p}(\Omega)$, it also converges to u in $L^p(\Omega)$, and hence in $L^p(\Omega')$ for all $\Omega' \subset \Omega$. Thus,

$$\nabla u_n \to \nabla u \qquad \text{in } (L^p(\Omega'))^N$$

is equivalent to

$$u_n \to u \qquad \text{in } W^{1,p}(\Omega').$$

This helps us understand the first part of the following remark.

Remark 3.2 The class $(\mathcal{S}_+)$ is contained in the classical class (S_+), defined in [6]. Furthermore, it is the case that the $p-$Laplacian belongs to the class $(\mathcal{S}_+)$.

The second part of Remark 3.2 can be deduced from [17], which showed that the $p-$Laplacian

$$-\Delta_p : W_0^{1,p}(\Omega') \to (W_0^{1,p}(\Omega'))^*$$

belongs to the class (S_+) for all bounded domains Ω' of $\mathbb{R}^N$.

We immediately deduce that $\mathcal{A}$ is continuous, as it can be written as the composition of the continuous maps, described as follows:

$$\begin{array}{ccccccccc} \mathcal{A} & : & W^{1,p}(\Omega) & \to & (L^p(\Omega))^N & \to & (L^{\frac{p}{p-1}}(\Omega))^N & \to & (W_0^{1,p}(\Omega))^* \\ & & u & \mapsto & \nabla u & \mapsto & N_A(\nabla u) & \mapsto & N_A(\nabla u). \end{array}$$

Here, the Nemytskii operator N_A, defined as

$$N_A(w(x)) := A(x, w(x)), \qquad \forall w \in (L^p(\Omega))^N,$$

is continuous because of condition (3.4). Note that the function $N_A(\nabla u)$ is in $(W_0^{1,p}(\Omega))^*$ in the following sense

$$\langle N_A(\nabla u), v\rangle = \int_\Omega N_A(\nabla u)\cdot \nabla v dx, \qquad \forall v \in W_0^{1,p}(\Omega).$$

With this observation, we see that the left-hand side of (3.1) is well-defined. However, the term $\mathcal{F}u$ in the right-hand side of (3.1) may not belong to $(W_0^{1,p}(\Omega))^*$ for some $u > 0$ in $W_0^{1,p}(\Omega)$ (we say here $u > 0$ because we are interested in positive solutions of (3.1)). One of the cases for this is that $f(x,\cdot,\xi)$, $x \in \Omega$, $x \in \mathbb{R}^N$, is undefined at 0^+. In this case, f and hence (3.1) are said to be singular. Solutions of singular problems will be understood in the sense of distributions, which is slightly different from the usual weak sense. We thus recall here these two kinds of solutions and sub-supersolutions as well.

3.2 Some concepts of solutions and sub-supersolutions

3.2.1 Weak sense

We define here the concepts of subsolution and supersolution when the domain of f is $\Omega \times \mathbb{R} \times \mathbb{R}^N$. One can see that these concepts are identical to those in [36, 37, 38] in the case that f is independent of the third variable and $f(\cdot, u) \in (W_0^{1,p}(\Omega))^*$, where u is either a sub- or a supersolution as defined below.

Definition 3.3 *The function $u \in W^{1,p}(\Omega)$ is a weak supersolution of (3.1) if, and only if:*

(i) $u|_{\partial\Omega} \geq 0$,

(ii) $\forall v \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$, $v \geq 0$,

$$\int_\Omega A(x, \nabla u) \cdot \nabla v dx \geq \int_\Omega f(x, u, \nabla u) v dx.$$

Definition 3.4 *The function $u \in W^{1,p}(\Omega)$ is a weak subsolution of (3.1) if, and only if:*

(i) $u|_{\partial\Omega} \leq 0$,

(ii) $\forall v \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$, $v \geq 0$,

$$\int_\Omega A(x, \nabla u) \cdot \nabla v dx \leq \int_\Omega f(x, u, \nabla u) v dx.$$

Remark 3.5 In the above definitions, the case that

$$\int_\Omega f(x, u, \nabla u) v dx = \pm\infty,$$

for some $v \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$, is permissible. However, we shall impose a growth condition on f so that the integral above is finite for all $v \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$.

A solution of (3.1) is defined as follows.

Definition 3.6 *The function $u \in W_0^{1,p}(\Omega)$ is a weak solution of (3.1) if, and only if:*

$$\int_\Omega A(x, \nabla u) \cdot \nabla v dx = \int_\Omega f(x, u, \nabla u) v dx$$

for all $v \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$.

This concept of solution was introduced in the well-known book [32]. However, its authors named this kind of a solution a generalized solution.

3.2.2 In the sense of distributions

In the case that the domain of f is $\Omega \times (0,\infty) \times \mathbb{R}^N$ and $f(x,\cdot,\xi)$ is allowed to blow up for $x \in \Omega$ and $\xi \in \mathbb{R}^N$ at 0^+, $\langle \mathcal{F}u, v\rangle$ is infinite or undefined for some $u, v \in W_0^{1,p}(\Omega)$. However, under some admissible conditions on f, this quantity is finite for all $v \in C_0^\infty(\Omega)$. This suggests the following definitions.

Definition 3.7 *The function $u \in W_{loc}^{1,p}(\Omega)$ is a subsolution of (3.1), in the sense of distributions, if, and only if:*

(i) $u > 0$ in Ω,

(ii) for all $v \in C_0^\infty(\Omega)$, $v \geq 0$,

$$\int_\Omega A(x, \nabla u) \cdot \nabla v dx \leq \int_\Omega f(x, u, \nabla u) v dx.$$

Definition 3.8 *The function $u \in W_{loc}^{1,p}(\Omega)$ is a supersolution of (3.1), in the sense of distributions, if, and only if:*

(i) $u > 0$ in Ω,

(ii) for all $v \in C_0^\infty(\Omega)$, $v \geq 0$,

$$\int_\Omega A(x, \nabla u) \cdot \nabla v dx \geq \int_\Omega f(x, u, \nabla u) v dx.$$

Since both sub- and supersolutions, defined above, are in $W_{loc}^{1,p}(\Omega)$, but might not belong to $W^{1,p}(\Omega)$, their traces on $\partial\Omega$ are not necessarily defined. However, this is not a problem, since we only need the traces of these functions defined on the boundaries of subdomains of Ω, instead of Ω.

Remark 3.9 The requirement of smoothness of the test function v may be relaxed because of the density of $C_0^\infty(\Omega')$ in $W_0^{1,p}(\Omega')$, for all open bounded and smooth domains Ω' of $\mathbb{R}^N$. More precisely, admissible test functions $v \geq 0$ may be required to belong to $W^{1,p}(\Omega) \cap L^\infty(\Omega)$ and have compact support.

3.3 Sub-supersolution theorems for nonsingular problems

We establish in this section two sub-supersolution theorems for the following particular form of (3.1):

$$\begin{cases} -\mathrm{div} A(x,\nabla u) &= f(x,u,\nabla u) & \text{in } \Omega, \\ u &= 0 & \text{on } \partial\Omega, \end{cases} \tag{3.7}$$

where f satisfies some growth conditions stated later in (3.8). Assume that (3.7) has k subsolutions $\underline{u}_1, \cdots, \underline{u}_k$, $k \geq 1$, and l supersolutions $\overline{u}_1, \cdots, \overline{u}_l$, $l \geq 1$, all of which belong to $C^1(\overline{\Omega})$, such that

$$\underline{u} := \max\{\underline{u}_1, \cdots, \underline{u}_k\} \leq \overline{u} := \min\{\overline{u}_1, \cdots, \overline{u}_l\} \quad \text{in } \Omega,$$

and there exist a function $a_3 \in L^{\frac{p}{p-1}}(\Omega)$ and a constant $b_3 > 0$ such that

$$|f(x,s,\xi)| \leq a_3(x) + b_3|\xi|^p, \tag{3.8}$$

for all $s \in [\underline{u}_0(x), \overline{u}_0(x)]$, $x \in \Omega$, where

$$\begin{aligned} \underline{u}_0 &:= \min\{\underline{u}_1, \cdots, \underline{u}_k\}, \\ \overline{u}_0 &:= \max\{\overline{u}_1, \cdots, \overline{u}_l\}. \end{aligned}$$

For each $(x,s) \in \Omega \times \mathbb{R}$, define

$$\gamma(x,s) := \begin{cases} \overline{u}(x), & s > \overline{u}(x), \\ s, & \underline{u}(x) \leq s \leq \overline{u}(x), \\ \underline{u}(x), & s < \underline{u}(x), \end{cases}$$

for $i = 1, \cdots, k$

$$\underline{\gamma}_i(x,s) := \begin{cases} \overline{u}(x), & s > \overline{u}(x), \\ s, & \underline{u}_i(x) \leq s \leq \overline{u}(x), \\ \underline{u}_i(x), & s < \underline{u}_i(x), \end{cases}$$

and for all $j = 1, \cdots, l$

$$\overline{\gamma}_j(x,s) := \begin{cases} \overline{u}_j(x), & s > \overline{u}_j(x), \\ s, & \underline{u}_j(x) \leq s \leq \overline{u}(x), \\ \underline{u}(x), & s < \underline{u}(x). \end{cases}$$

Then for all $u \in W^{1,p}(\Omega)$, the functions

$$\begin{aligned} x &\mapsto \gamma(x, u(x)), \\ x &\mapsto \underline{\gamma}_i(x, u(x)), \qquad i = 1, \cdots, k, \end{aligned}$$

$$x \mapsto \overline{\gamma}_j(x, u(x)), \quad j = 1, \cdots, l$$

belong to $W^{1,p}(\Omega)$. For each $n \in \mathbb{N}$, let

$$h_n : \mathbb{R}^N \to \mathbb{R}^N$$

be defined as

$$h_n(\xi) := \begin{cases} \xi, & |\xi| \le n, \\ \frac{n}{|\xi|}\xi, & |\xi| > n, \end{cases} \qquad \text{for all } \xi \in \mathbb{R}^N.$$

We now consider the auxiliary problem

$$\begin{cases} -\mathrm{div}(A(x, \nabla u)) &= \tilde{f}_n(x, u) & \text{in } \Omega, \\ u &= 0 & \text{on } \partial\Omega, \end{cases} \tag{3.9}$$

where

$$\begin{aligned} \tilde{f}_n(x, u) &:= f(x, \gamma(x, u), h_n(\nabla\gamma(x, u))) \\ &+ \sum_{i=1}^{k} |f(x, \gamma(x, u), h_n(\nabla\gamma(x, u))) - f(x, \underline{\gamma}_i(x, u), h_n(\nabla\underline{\gamma}_i(x, u)))| \\ &- \sum_{j=1}^{l} |f(x, \gamma(x, u), h_n(\nabla\gamma(x, u))) - f(x, \overline{\gamma}_j(x, u), h_n(\nabla\overline{\gamma}_j(x, u)))|. \end{aligned}$$

For each $n \ge 1$, define

$$\mathcal{F}_n : W_0^{1,p}(\Omega) \to (W_0^{1,p}(\Omega))^*$$

as

$$\langle \mathcal{F}_n(u), v \rangle := \int_\Omega \tilde{f}_n(x, u) v dx \qquad \forall u, v \in W_0^{1,p}(\Omega).$$

Lemma 3.10 *$\mathcal{F}_n$ is demicontinuous for all $n \ge 1$. That is, if $u_m \to u$ in $W_0^{1,p}(\Omega)$, then*

$$\lim_{m\to\infty} \langle \mathcal{F}_n(u_m), v \rangle = \langle \mathcal{F}_n(u), v \rangle,$$

for each $v \in W_0^{1,p}(\Omega)$.

Proof : The lemma is proved by using (3.8), the boundedness of h_n, $n \ge 1$, the continuity of γ, $\underline{\gamma}_i$ and $\overline{\gamma}_j$ from $W^{1,p}(\Omega)$ to $W^{1,p}(\Omega)$, $1 \le i \le k$, $1 \le j \le l$, and Lebesgue's dominated convergence theorem.

Lemma 3.11 *For all $n \ge 1$, $\mathcal{A} - \mathcal{F}_n$ is of class (S_+).*

Proof : Assuming that

$$u_m \rightharpoonup u \quad \text{in } W_0^{1,p}(\Omega),$$

$$u_m \quad \to \quad u \qquad \text{in } L^p(\Omega),$$

we may use (3.8), noting that all functions $\gamma(x,s)$, $\underline{\gamma}_i(x,s)$ and $\overline{\gamma}_j(x,s)$, $1 \le i \le k$, $1 \le j \le l$ have their range in the interval $[\underline{u}_0(x), \overline{u}_0(x)]$ for a.e. $x \in \Omega$ and all $s \in \mathbb{R}$, the boundedness of h_n and Hölder's inequality to show that

$$\lim_{m\to\infty} |\langle \mathcal{F}_n(u_m), u_m - u\rangle| = 0.$$

Hence, the fact that

$$\limsup_{m\to\infty} \langle \mathcal{A}(u_m) - \mathcal{F}_n(u_m), u_m - u\rangle \le 0$$

implies that

$$\limsup_{m\to\infty} \langle \mathcal{A}(u_m), u_m - u\rangle \le 0,$$

and hence $u_m \to u$ in $W_0^{1,p}(\Omega)$ because $\mathcal{A}$ is of class $(\mathcal{S}_+)$, which is contained is the class (S_+).

Fix $n \ge 1$. Let $\mathcal{G}$ be the family of demicontinuous operators of class (S_+)

$$G : \overline{B_{R_n}} \to (W_0^{1,p}(\Omega))^*,$$

where R_n is a large positive number and

$$B_{R_n} = \{u \in W_0^{1,p}(\Omega) : \|u\| < R_n\}.$$

Let $\mathcal{H}$ be the class of affine homotopies in $\mathcal{G}$ and let J be the dual mapping from $W_0^{1,p}(\Omega)$ to $(W_0^{1,p}(\Omega))^*$. According to Theorem 4 in [6], there exists one and only one degree function deg on $\mathcal{G}$ which is normalized by the map J and invariant under $\mathcal{H}$.

Using (3.6), we see that

$$\begin{aligned} H_1 : [0,1] \times \overline{B_{R_n}} &\longrightarrow (W_0^{1,p}(\Omega))^* \\ H_1(t,u) &= (1-t)\mathcal{A}(u) + tJ(u) \end{aligned}$$

is an affine homotopy when R_n is large enough. This yields

$$\deg(\mathcal{A}, B_{R_n}, 0) = 1.$$

On the other hand, the map

$$H_2 : [0,1] \times \overline{B_{R_n}} \to (W_0^{1,p}(\Omega))^*$$

$$H_2(t, u) \quad = \quad (1-t)\mathcal{A}(u) + t\mathcal{F}_n(u)$$

is also in $\mathcal{H}$. In fact, if we assume there are $t \in [0,1]$ and $u \in \partial B_{R_n}$ such that

$$H_2(t, u) = 0,$$

or equivalently,

$$\int_\Omega A(x, \nabla u) \cdot \nabla v dx - t \int_\Omega \tilde{f}_n(x, u) v dx = 0, \tag{3.10}$$

for all $v \in W_0^{1,p}(\Omega)$, then, using $v = u$ as the test function in (3.10), together with the fact that $|h_n|$ is bounded by n, we have

$$\int_\Omega b_2 |\nabla u|^p dx \leq \int_\Omega [a_2 + ((1+2k+2l)a_3 + b_3(1+2k+2l)n^p)|u|] dx,$$

which is impossible because R_n was chosen large. By Theorem 4 in [6],

$$\deg(H_2(t, \cdot), B_{R_n}, 0)$$

is well-defined and invariant as t varies in $[0,1]$. This gives

$$\deg(\mathcal{A} - \mathcal{F}_n, B_{R_n}, 0) = \deg(\mathcal{A}, B_{R_n}, 0) = 1,$$

and therefore the equation $\mathcal{A} - \mathcal{F}_n = 0$ has a solution $u_n \in W_0^{1,p}(\Omega)$.

Remark 3.12 It is not surprising that u_n, obtained in the previous paragraph, is essentially bounded, although $f(\cdot, u_n, \nabla u_n)$ is bounded in terms of $|\nabla u_n|^p$ (see condition (3.8)). In earlier work of Ladyzhenskaya and Ural'tseva [32] such terms are not included. The main reason yielding L^∞ bounds is that we have replaced f by $\tilde{f}_n$, which satisfies the Leray-Lions conditions (see [32]), because of the boundedness of h_n.

Lemma 3.13 *For each $n \geq 1$, u_n is essentially bounded.*

Proof : Since all functions $\gamma(x, w)$, $\underline{\gamma}_i(x, w)$, $\overline{\gamma}_j(x, w)$, $1 \leq i \leq k$, $1 \leq j \leq l$, are uniformly bounded with respect to $x \in \Omega$, $w \in W^{1,p}(\Omega)$ and since $h_n(\xi)$ is uniformly bounded with respect to $\xi \in \mathbb{R}^N$, $|\tilde{f}_n(x, w)|$ is dominated by

$$(1+2k+2l)a_3(x) + M_n$$

for a.e. $x \in \Omega$, all $w \in W^{1,p}(\Omega)$. It follows that u_n satisfies

$$|-\operatorname{div} A(x, \nabla u_n)| \leq (1+2k+2l)a_3(x) + M_n,$$

where k and l are, respectively, the number of sub- and supersolutions.

Using the monotonicity condition satisfied by A (condition (3.5)) and the weak comparison principle, we conclude that

$$\mu_n(x) \leq u_n(x) \leq \rho_n(x) \qquad \text{for a.e. } x \in \Omega,$$

where ρ_n is the solution of

$$\begin{cases} -\mathrm{div}A(x, \nabla\rho_n) &= (1+2k+2l)a_3(x) + M_n \quad \text{in } \Omega, \\ \rho_n &= 0 \quad \text{on } \partial\Omega, \end{cases}$$

and μ_n is the solution of

$$\begin{cases} -\mathrm{div}A(x, \nabla\mu_n) &= -(1+2k+2l)a_3(x) - M_n \quad \text{in } \Omega, \\ \mu_n &= 0 \quad \text{on } \partial\Omega, \end{cases}$$

Since ρ_n and μ_n are bounded (see [32]), so is u_n.

We make the convention that the assertion $w \in [w_1, w_2]$, where w, w_1 and w_2 are measurable and defined on Ω, is to be understood to mean

$$w_1(x) \leq w(x) \leq w_2(x),$$

for a.e. $x \in \Omega$.

Lemma 3.14 *For all* $n \geq \max\{\|\nabla \underline{u}_i\|_{L^\infty(\Omega)}, \|\nabla \overline{u}_j\|_{L^\infty(\Omega)} : 1 \leq i \leq k, 1 \leq j \leq l\}$, $u_n \in [\underline{u}, \overline{u}]$.

Proof : Fix $i \in \{1, \cdots, k\}$. Since $(u_n - \underline{u}_i)^- \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ (see Lemma 3.13 for the boundedness of $(u_n - \underline{u}_i)$), it is an admissible test function for (3.9). We have

$$\begin{aligned} &\int_\Omega A(x, \nabla u_n) \cdot \nabla(u_n - \underline{u}_i)^- dx \\ &\quad = \int_\Omega \tilde{f}_n(x, u_n)(u_n - \underline{u}_i)^- dx \\ &\quad \geq \int_\Omega [f(x, \underline{u}, \nabla\underline{u}) + |f(x, \underline{u}, \nabla\underline{u}) - f(x, \underline{u}_i, \nabla\underline{u}_i)|](u_n - \underline{u}_i)^- dx \\ &\quad \geq \int_\Omega f(x, \underline{u}_i, \nabla\underline{u}_i)(u_n - \underline{u}_i)^- dx \\ &\quad \geq \int_\Omega A(x, \nabla\underline{u}_i)\nabla(u_n - \underline{u}_i)^- dx, \end{aligned}$$

where the first inequality follows from the fact that

$$n \geq \max\{\|\nabla \underline{u}_i\|_{L^\infty(\Omega)}, \|\nabla \overline{u}_j\|_{L^\infty(\Omega)} : 1 \leq i \leq k, 1 \leq j \leq l\}.$$

Therefore,

$$\int_\Omega [A(x, \nabla u_n) - A(x, \nabla\underline{u}_i)] \cdot \nabla(u_n - \underline{u}_i)^- dx \geq 0,$$

which yields $(u_n - \underline{u}_i)^- = 0$ and, hence, $u_n \geq \underline{u}_i$, by using the formula

$$\nabla(u_n - \underline{u}_i)^- = \begin{cases} -\nabla(u_n - \underline{u}_i) & u_n - \underline{u}_i \leq 0, \\ 0 & u_n - \underline{u}_i \geq 0, \end{cases}$$

and the strict monotonicity of A.

Similarly, using the test function $(u_n - \overline{u}_j)^+$ in (3.9) gives $u \leq \overline{u}_j$ for any $j \in \{1, \cdots, l\}$.

Without loss of generality, we may assume that Lemma 3.14 is valid for all $n \geq 1$. Thus, u_n solves

$$\begin{cases} -\mathrm{div}(A(x, \nabla u_n) &= f(x, u_n, h_n(\nabla u_n)) \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{cases} \tag{3.11}$$

for all $n \geq 1$.

Lemma 3.15 *$\{u_n\}_{n\in\mathbb{N}}$ is bounded in $W_0^{1,p}(\Omega)$.*

Proof : By Lemma 3.14, the sequence $\{\|u_n\|_{L^\infty(\Omega)}\}_{n\in\mathbb{N}}$ is uniformly bounded by the maximum of $\|\overline{u}\|_{L^\infty(\Omega)}$ and $\|\underline{u}\|_{L^\infty(\Omega)}$. Thus, the function

$$v_t = e^{tu_n^2} u_n,$$

whose $i - th$ partial derivative is

$$\frac{\partial v_t}{\partial x_i} = e^{tu_n^2}(1 + 2tu_n^2)\frac{\partial u_n}{\partial x_i}, \qquad i \in \{1, 2, \cdots, N\},$$

belongs to $W_0^{1,p}(\Omega)$, for any positive real number t. Using v_t as the test function in (3.11), we have

$$\int_\Omega A(x, \nabla u_n) \cdot \nabla v_t dx = \int_\Omega f(x, u_n, h_n(\nabla u_n)) v_t dx,$$

which is equivalent to

$$\int_\Omega e^{tu_n^2}(1 + 2tu_n^2) A(x, \nabla u_n) \cdot \nabla u_n dx = \int_\Omega e^{tu_n^2} f(x, u_n, h_n(\nabla u_n)) u_n dx.$$

This, together with conditions (3.6) and (3.8), shows that

$$\int_\Omega e^{tu_n^2}(1 + 2tu_n^2)(b_2|\nabla u_n|^p - a_2) dx \leq \int_\Omega e^{tu_n^2}(a_3 + b_3|\nabla u_n|^p)|u_n| dx.$$

Again, since, $\|u_n\|_{L^\infty(\Omega)}$ is uniformly bounded, we can find a constant C such that

$$\int_\Omega b_2 e^{tu_n^2}(1 + 2tu_n^2)|\nabla u_n|^p dx \quad \leq \quad C + b_3 \int_\Omega e^{tu_n^2}|\nabla u_n|^p |u_n| dx$$

$$\leq \quad C + b_3 \int_\Omega e^{tu_n^2} |\nabla u_n|^p (\frac{\epsilon}{4} + \frac{u_n^2}{\epsilon}) dx,$$

where ϵ is an arbitrary positive number. We now choose ϵ so small that

$$\frac{b_3 \epsilon}{4} < \frac{b_2}{2}$$

and then choose t so that

$$2tb_2 = \frac{b_3}{\epsilon}$$

and obtain

$$\int_\Omega \frac{b_2}{2} e^{tu_n^2} |\nabla u_n|^p dx \leq C.$$

It follows that $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $W_0^{1,p}(\Omega)$.

Since $W_0^{1,p}(\Omega)$ is reflexive, we can find a function $u \in W_0^{1,p}(\Omega)$ and a subsequence of $\{u_n\}_{n\in\mathbb{N}}$, still denoted by $\{u_n\}_{n\in\mathbb{N}}$, such that

$$\begin{array}{lcll} u_n & \rightharpoonup & u & \text{in } W_0^{1,p}(\Omega), \\ u_n & \to & u & \text{in } L^p(\Omega), \\ u_n & \to & u & \text{a.e. in } \Omega. \end{array}$$

Here, we have used the compact embedding from $W_0^{1,p}(\Omega)$ to $L^p(\Omega)$ and Theorem 1.Q in [54].

Lemma 3.16 *The sequence $\{u_n\}_{n\in\mathbb{N}}$ converges to u in $W_0^{1,p}(\Omega)$.*

Proof : For $n \geq 1$, using

$$e^{t(u_n-u)^2}(u_n - u) \in W_0^{1,p}(\Omega)$$

as a test function in (3.11) and using condition (3.6), we have

$$\begin{aligned} &\int_\Omega (1 + 2t(u_n - u)^2) e^{t(u_n-u)^2} A(x, \nabla u_n) \cdot \nabla(u_n - u) dx \\ &\qquad = \int_\Omega e^{t(u_n-u)^2} f(x, u_n, h_n(\nabla u_n))(u_n - u) dx \\ &\qquad \leq \int_\Omega e^{t(u_n-u)^2} (a_3 + b_3 |\nabla u_n|^p) |u_n - u| dx \\ &\qquad = \int_\Omega e^{t(u_n-u)^2} a_3 |u_n - u| dx + \int_\Omega b_3 e^{t(u_n-u)^2} |\nabla u_n|^p |u_n - u| dx. \end{aligned}$$

Using condition (3.6), we see that

$$\int_\Omega b_3 e^{t(u_n-u)^2} |\nabla u_n|^p |u_n - u| dx$$

$$\leq \int_\Omega b_3 e^{t(u_n-u)^2} |\nabla u_n|^p \left(\frac{\epsilon}{2} + \frac{1}{2\epsilon}(u_n-u)^2\right) dx$$
$$\leq \int_\Omega \frac{b_3}{b_2} e^{t(u_n-u)^2} (a_2 + A(x,\nabla u_n)\cdot \nabla u_n) \left(\frac{\epsilon}{2} + \frac{1}{2\epsilon}(u_n-u)^2\right) dx,$$

and consequently,

$$\begin{aligned}
&\int_\Omega (1+2t(u_n-u)^2) e^{t(u_n-u)^2} A(x,\nabla u_n)\cdot\nabla(u_n-u)dx \\
&\quad\leq \int_\Omega e^{t(u_n-u)^2} a_3 |u_n-u| dx \\
&\qquad + \frac{\epsilon b_3}{2b_2}\int_\Omega e^{t(u_n-u)^2} a_2 dx + \frac{b_3}{2\epsilon b_2}\int_\Omega e^{t(u_n-u)^2} a_2 (u_n-u)^2 dx \\
&\qquad + \frac{\epsilon b_3}{2b_2}\int_\Omega e^{t(u_n-u)^2} A(x,\nabla u_n)\cdot\nabla(u_n-u)dx \\
&\qquad + \frac{\epsilon b_3}{2b_2}\int_\Omega e^{t(u_n-u)^2} A(x,\nabla u_n)\cdot\nabla u dx \\
&\qquad + \frac{b_3}{2\epsilon b_2}\int_\Omega e^{t(u_n-u)^2} (u_n-u)^2 A(x,\nabla u_n)\cdot\nabla(u_n-u)dx \\
&\qquad + \frac{b_3}{2\epsilon b_2}\int_\Omega e^{t(u_n-u)^2} (u_n-u)^2 A(x,\nabla u_n)\cdot\nabla u dx.
\end{aligned}$$

We next choose

$$t = \frac{b_3}{4\epsilon b_2}$$

and then combine the left-hand side of the above inequality with the fourth and sixth summands of the right-hand side to obtain

$$\begin{aligned}
&\left(1 - \frac{\epsilon b_3}{2b_2}\right)\int_\Omega e^{t(u_n-u)^2} A(x,\nabla u_n)\cdot\nabla(u_n-u)dx \\
&\quad\leq \int_\Omega e^{t(u_n-u)^2} a_3 |u_n-u| dx \\
&\qquad + \frac{\epsilon b_3}{2b_2}\int_\Omega e^{t(u_n-u)^2} a_2 dx + \frac{b_3}{2\epsilon b_2}\int_\Omega e^{t(u_n-u)^2} a_2 (u_n-u)^2 dx \\
&\qquad + \frac{\epsilon b_3}{2b_2}\int_\Omega e^{t(u_n-u)^2} A(x,\nabla u_n)\cdot\nabla u dx \\
&\qquad + \frac{b_3}{2\epsilon b_2}\int_\Omega e^{t(u_n-u)^2} (u_n-u)^2 A(x,\nabla u_n)\cdot\nabla u dx.
\end{aligned}$$

Using Hölder's inequality, Lebesgue's dominated convergence theorem, the boundedness of $\{u_n\}_{n\in\mathbb{N}}$ in $W_0^{1,p}(\Omega)$, condition (3.4), and letting n tend to ∞, we have

$$\limsup_{n\to\infty}\left(1 - \frac{\epsilon b_3}{2b_2}\right)\int_\Omega e^{t(u_n-u)^2} A(x,\nabla u_n)\cdot\nabla(u_n-u)dx \leq \epsilon C, \tag{3.12}$$

where C is a positive number depending on a_i, b_i, $i = 1,2,3$, and $\sup\{\|u_n\| : n\in\mathbb{N}\}$.

On the other hand, the quantity

$$\left|\int_\Omega (e^{t(u_n-u)^2}-1)A(x,\nabla u)\cdot\nabla(u_n-u)dx\right| \tag{3.13}$$

is bounded from above by

$$\left(\int_\Omega |(e^{t(u_n-u)^2}-1)A(x,\nabla u)|^{\frac{p}{p-1}}dx\right)^{\frac{p-1}{p}}\left(\int_\Omega |\nabla(u_n-u)|^p dx\right)^{\frac{1}{p}},$$

which tends to 0 as $n\to\infty$ because of the dominated convergence theorem and the boundedness of $\{u_n-u\}_{n\in\mathbb{N}}$ in both spaces $L^\infty(\Omega)$ and $W_0^{1,p}(\Omega)$. Thus, the quantity in (3.13) converges to 0 and, therefore,

$$\begin{aligned}\lim_{n\to\infty}\int_\Omega & e^{t(u_n-u)^2}A(x,\nabla u)\cdot\nabla(u_n-u)dx \\ &= \lim_{n\to\infty}\int_\Omega A(x,\nabla u)\cdot\nabla(u_n-u)dx = 0.\end{aligned}$$

The last equality holds because $u_n\rightharpoonup u$ in $W_0^{1,p}(\Omega)$. This, together with (3.12), implies that

$$\begin{aligned}\limsup_{n\to\infty}\left(1-\frac{\epsilon b_3}{2b_2}\right)\int_\Omega e^{t(u_n-u)^2}[A(x,\nabla u_n)-A(x,\nabla u)]\cdot\nabla(u_n-u)dx \\ \leq \epsilon C.\end{aligned}$$

Since $e^{t(u_n-u)^2}>1$ for all $t>0,\ n\geq 1$ and a.e. $x\in\Omega$,

$$\limsup_{n\to\infty}\left(1-\frac{\epsilon b_3}{2b_2}\right)\int_\Omega [A(x,\nabla u_n)-A(x,\nabla u)]\cdot\nabla(u_n-u)dx\leq\epsilon C.$$

Letting $\epsilon\to 0^+$, we obtain

$$\limsup_{n\to\infty}\int_\Omega A(x,\nabla u_n)\cdot\nabla(u_n-u)dx\leq 0,$$

and hence $u_n\to u$ in $W_0^{1,p}(\Omega)$ because $\mathcal{A}$ is of class (S_+).

Applying Theorem 1.Q in [54], we can find a function $w\in L^p(\Omega)$ and a subsequence of $\{u_n\}_{n\in\mathbb{N}}$, still denoted by $\{u_n\}_{n\in\mathbb{N}}$, such that

$$\nabla u_n\to\nabla u$$

and

$$|\nabla u_n(x)|\leq w(x),$$

for a.e. $x\in\Omega$. Fix $v\in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$. Using the inequality above and Lebesgue's dominated convergence theorem and letting $n\to\infty$ in the equation

$$\int_\Omega A(x,\nabla u_n)\cdot\nabla v dx=\int_\Omega f(x,u_n,h_n(\nabla u_n))v dx,$$

we see that u is a weak solution of (3.7). We have proved the following theorem.

Theorem 3.17 *Assume that problem (3.7) has k subsolutions $\underline{u}_i$, $i = 1, \cdots, k$, and l supersolutions $\overline{u}_1, \cdots, \overline{u}_l$, $k, l \geq 1$, all of which belong to $C^1(\overline{\Omega})$, such that*

$$\underline{u} := \max\{\underline{u}_i : i = 1, \cdots, k\} \leq \overline{u} := \min\{\overline{u}_j : j = 1, \cdots, l\} \qquad \text{in } \Omega.$$

Assume further that there exist a function $a_3 \in L^{\frac{p}{p-1}}(\Omega)$ and a constant $b_3 > 0$ such that

$$|f(x, s, \xi)| \leq a_3(x) + b_3|\xi|^p,$$

for all $s \in [\underline{u}_0(x), \overline{u}_0(x)]$, for a.e. $x \in \Omega$, where

$$\begin{aligned} \underline{u}_0 &:= \min\{\underline{u}_1, \cdots, \underline{u}_k\}, \\ \overline{u}_0 &:= \max\{\overline{u}_1, \cdots, \overline{u}_l\}. \end{aligned}$$

Then (3.7) has a solution $u \in W_0^{1,p}(\Omega)$, satisfying

$$\underline{u}(x) \leq u(x) \leq \overline{u}(x) \qquad \text{in } \Omega.$$

Remark 3.18 The requirement that all sub-supersolutions in Theorem 3.17 belong to $C^1(\overline{\Omega})$ can be replaced by the weaker condition that they are in $W^{1,\infty}(\Omega)$ because we only use their boundedness in $W^{1,\infty}(\Omega)$ to estimiate inequalities globally, not at any single point in Ω.

Remark 3.19 Although the solution u obtained in Theorem 3.17 is a subsolution or a supersolution of (3.7), we cannot use it as a subsolution or a supersolution when applying this theorem because u is not in $C^1(\overline{\Omega})$. Since we want to consider u as a subsolution and a supersolution later in the next section, we wish to employ some regularity results to study the smoothness of u. However, it is known from regularity theory that, under some additional conditions on A so that (3.7) is uniformly elliptic, $u \in C^{1,\beta}(\Omega)$, $\beta > 0$, because u is bounded by $\underline{u}$ and $\overline{u}$ (see [32, 40]). This does not imply that $u \in C^1(\overline{\Omega})$. We thus replace the condition that all given sub-supersolutions are in $C^1(\overline{\Omega})$ by the weaker one that they belong to $C^1(\Omega)$. That this may be accomplished follows by using approximation techniques.

From now on, assume A that satisfies some additional conditions so that Lieberman's regularity results [40] hold; namely, A is differentiable (except possible at $\xi = 0$) and there exist $b_4, b_5 > 0$ such that

$$\sum_{i,j=1}^{N} a_{ij}(x, \xi)\eta_i\eta_j \geq b_4|\xi|^{p-2}|\eta|^2, \qquad \forall \eta \in \mathbb{R}^N \tag{3.14}$$

and

$$|a_{ij}(x,\xi)| \leq b_5|\xi|^{p-2} \qquad \forall 1 \leq i,j \leq N, \tag{3.15}$$

for $x \in \Omega, \xi \in \mathbb{R}^N \setminus \{0\}$. Here, a_{ij} denotes $\frac{\partial A_i(x,\xi)}{\partial \xi_j}$ and A_i is the $i-$th component of A, $1 \leq i,j \leq N$.

Theorem 3.20 *Assume that problem (3.7) has k subsolutions $\underline{u}_i$, $i = 1,\cdots,k$, and l supersolutions $\overline{u}_1,\cdots,\overline{u}_l$, $k,l \geq 1$, all of which belong to $C^1(\Omega)$, such that*

$$\underline{u} := \max\{\underline{u}_i : i = 1,\cdots,k\} \leq \overline{u} := \min\{\overline{u}_j : j = 1,\cdots,l\} \qquad \text{in } \Omega.$$

Assume further that there exist a function $a_3 \in L^{\frac{p}{p-1}}(\Omega)$ and a constant $b_3 > 0$ such that

$$|f(x,s,\xi)| \leq a_3(x) + b_3|\xi|^p,$$

for all $s \in [\underline{u}_0(x), \overline{u}_0(x)]$, for a.e. $x \in \Omega$, where

$$\begin{aligned} \underline{u}_0 &:= \min\{\underline{u}_1,\cdots,\underline{u}_k\}, \\ \overline{u}_0 &:= \max\{\overline{u}_1,\cdots,\overline{u}_l\}. \end{aligned}$$

Then (3.7) has a solution $u \in C^{1,\beta}(\Omega)$, for some $\beta > 0$, satisfying

$$\underline{u}(x) \leq u(x) \leq \overline{u}(x) \qquad \text{in } \Omega.$$

Proof : Let $\{\Omega_n\}_{n\in\mathbb{N}}$ be the sequence of smooth subdomains of Ω such that

$$\overline{\Omega}_n \subset \Omega_{n+1}, \qquad n \geq 1$$

and

$$\bigcup_{n\in\mathbb{N}} = \Omega.$$

For any $n \geq 1$, the problem

$$\begin{cases} -\text{div}A(x,\nabla(v+\underline{u})) &= f(x,v+\underline{u},\nabla(v+\underline{u})) \quad \text{in } \Omega_n, \\ \qquad v &= 0 \qquad\qquad\qquad\qquad\qquad \text{on } \partial\Omega_n \end{cases} \tag{3.16}$$

has $\underline{u}_i - \underline{u}$ and $\overline{u}_j - \underline{u}$ as subsolution and supersolution, respectively, $1 \leq i \leq k$, $1 \leq j \leq l$. Although these functions do not belong to $C^1(\overline{\Omega}_n)$, we might apply Theorem 3.17 to find a weak solution v_n, whose range is in $[0, \overline{u}-\underline{u}]$, of (3.16) because of the observation in Remark 3.18. Define $v_n = 0$ on $\Omega \setminus \Omega_n$. Employing the test function $e^{tv_n^2}v_n \in W_0^{1,p}(\Omega_n) \cap L^\infty(\Omega_n)$ for

(3.16) and repeating the arguments in Lemma 3.15, we can show that $\{v_n\}_{n\in\mathbb{N}}$ is bounded in $W_0^{1,p}(\Omega)$. Thus, for all $n \in \mathbb{N}$, if u_n is defined as $v_n + \underline{u}$, then u_n solves

$$\begin{cases} -\mathrm{div} A(x, \nabla u_n) &= f(x, u_n, \nabla u_n) \quad \text{in } \Omega_n, \\ u_n &= \underline{u} \quad \text{in } \Omega \setminus \Omega_n, \end{cases} \tag{3.17}$$

and the sequence $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $W_0^{1,p}(\Omega)$.

Since $W_0^{1,p}(\Omega)$ is reflexive, we can assume that $\{u_n\}_{n\in\mathbb{N}}$ weakly converges to u in $W_0^{1,p}(\Omega)$ and converges to u a.e. in Ω. We have the lemma.

Lemma 3.21 *Let K be the closure of an open subset of Ω. Then the sequence $\{u_n\}_{n\in\mathbb{N}}$ converges to u in $W^{1,p}(K)$.*

Proof : Let φ be a nonnegative function in $C_0^\infty(\Omega)$ with $\varphi = 1$ in K and Ω' be the support of φ. Without loss of generality, assume that $\Omega' \subset \Omega_1$. Fix $n \in \mathbb{N}$. Use

$$w_t = e^{t(u_n-u)^2}\varphi^p(u_n - u) \in W_0^{1,p}(\Omega_n) \cap L^\infty(\Omega_n),$$

whose gradient is

$$\nabla w_t = e^{t(u_n-u)^2}\varphi^p(2t(u_n-u)^2+1)\nabla(u_n-u) + pe^{t(u_n-u)^2}\varphi^{p-1}\nabla\varphi,$$

where $t > 0$ will be chosen later, as the test function for (3.17) to get

$$\begin{aligned} &\int_\Omega e^{t(u_n-u)^2}\varphi^p[2t(u_n-u)^2+1]A(x,\nabla u_n)\cdot\nabla(u_n-u)dx \\ &\quad +p\int_\Omega e^{t(u_n-u)^2}(u_n-u)\varphi^{p-1}A(x,\nabla u_n)\cdot\nabla\varphi dx \\ &= \int_\Omega f(x,u_n,\nabla u_n)e^{t(u_n-u)^2}\varphi^p(u_n-u)dx \\ &\le \int_\Omega |a_3|e^{t(u_n-u)^2}\varphi^p|u_n-u|dx + b_3\int_\Omega e^{t(u_n-u)^2}\varphi^p|u_n-u||\nabla u_n|^p dx. \end{aligned}$$

Let ϵ be an arbitrary positive number and define

$$c_{n,t} := \int_\Omega |a_3|e^{t(u_n-u)^2}\varphi^p|u_n-u|dx,$$

which converges to 0 as n tends to ∞. Then the last quantity above is bounded from above by

$$\begin{aligned} &c_{n,t} + b_3\int_\Omega e^{t(u_n-u)^2}\varphi^p\left(\frac{\epsilon}{2}+\frac{(u_n-u)^2}{2\epsilon}\right)|\nabla u_n|^p dx \\ &\le c_{n,t} + \frac{b_3}{b_2}\int_\Omega e^{t(u_n-u)^2}\varphi^p\left(\frac{\epsilon}{2}+\frac{(u_n-u)^2}{2\epsilon}\right)(A(x,\nabla u_n)\cdot\nabla u_n + a_2)dx. \end{aligned}$$

Consequently,

$$\int_\Omega e^{t(u_n-u)^2}\varphi^p[2t(u_n-u)^2+1]A(x,\nabla u_n)\cdot\nabla(u_n-u)dx$$

$$
\begin{aligned}
&\leq c_{n,t} + \frac{b_3}{b_2}\int_\Omega e^{t(u_n-u)^2}\varphi^p\left(\frac{\epsilon}{2}+\frac{(u_n-u)^2}{2\epsilon}\right)A(x,\nabla u_n)\cdot\nabla u dx\\
&+\frac{b_3}{b_2}\int_\Omega e^{t(u_n-u)^2}\varphi^p\left(\frac{\epsilon}{2}+\frac{(u_n-u)^2}{2\epsilon}\right)A(x,\nabla u_n)\cdot\nabla(u_n-u)dx,\\
&+\frac{b_3}{b_2}\int_\Omega e^{t(u_n-u)^2}\varphi^p\left(\frac{\epsilon}{2}+\frac{(u_n-u)^2}{2\epsilon}\right)a_2 dx.
\end{aligned}
$$

We now choose $t = \frac{b_3}{4\epsilon b_2}$ so that the inequalities above imply

$$
\limsup_{n\to\infty}\left(1-\frac{\epsilon b_3}{2b_2}\right)\int_\Omega e^{t(u_n-u)^2}\varphi^p A(x,\nabla u_n)\cdot\nabla(u_n-u)dx \leq \epsilon C,
$$

where C is a positive constant depending only on $\varphi, a_i, b_i, i = 1, 2, 3$, and $\sup\{\|u_n\| : n\in\mathbb{N}\}$.

As in the proof of Lemma 3.16, we may use the boundedness of $\{u_n\}_{n\in\mathbb{N}}$ in $L^\infty(\Omega)$, Hölder's inequality, and the dominated convergence theorem to see that

$$
\lim_{n\to\infty}\int_\Omega (e^{t(u_n-u)^2}-1)\varphi^p A(x,\nabla u)\cdot\nabla(u_n-u)dx = 0.
$$

Therefore,

$$
\begin{aligned}
&\lim_{n\to\infty}\int_\Omega e^{t(u_n-u)^2}\varphi^p A(x,\nabla u)\cdot\nabla(u_n-u)dx\\
&\quad=\lim_{n\to\infty}\int_\Omega \varphi^p A(x,\nabla u)\cdot\nabla(u_n-u)dx = 0.
\end{aligned}
$$

The last equality is obtained from the weak convergence of $\{u_n\}_{n\in\mathbb{N}}$ to u in $W^{1,p}(\Omega')$. Hence,

$$
\begin{aligned}
&\limsup_{n\to\infty}\left(1-\frac{\epsilon b_3}{2b_2}\right)\int_\Omega e^{t(u_n-u)^2}\varphi^p[A(x,\nabla u_n)-A(x,\nabla u)]\cdot\nabla(u_n-u)dx\\
&\leq \epsilon C.
\end{aligned}
$$

Noting that the integrand in the inequality above is nonnegative, $e^{t(u_n-u)^2}\geq 1$ in Ω and $\varphi = 1$ in K, we have

$$
\limsup_{n\to\infty}\left(1-\frac{\epsilon b_3}{2b_2}\right)\int_K [A(x,\nabla u_n)-A(x,\nabla u)]\cdot\nabla(u_n-u)dx \leq \epsilon C,
$$

Letting $\epsilon\to 0^+$ in the inequality above, we conclude

$$
\limsup_{n\to\infty}\int_K [A(x,\nabla u_n)-A(x,\nabla u)]\cdot\nabla(u_n-u)dx \leq 0.
$$

Since $\mathcal{A}$ is of class $(\mathcal{S}_+)$, $u_n\to u$ in $W^{1,p}(K)$.

This lemma helps us to see that

$$
\int_\Omega A(x,\nabla u)\cdot\nabla\phi dx = \int_\Omega f(x,u,\nabla u)\phi dx
$$

for all $\phi\in C_0^\infty(\Omega)$. This is also true for ϕ in $W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$ because of the density of $C_0^\infty(\Omega)$ in $W_0^{1,p}(\Omega)$ and condition (3.8).

On the other hand, Theorem 1.7 in [40] can be used to deduce that $u\in C^{1,\beta}(\Omega)$ for some $\beta\in(0,1)$.

3.4 Extremal solutions

We begin this section by defining the concepts of minimal and maximal solutions between a pair of functions.

Definition 3.22 *Let $\underline{u}$ and $\overline{u}$ be two measurable functions defined on Ω such that $\underline{u}(x) \leq \overline{u}(x)$ a.e. in Ω. The function u is said to be the minimal (or maximal) solution of (3.7) with respect to the pair $\underline{u}$ and $\overline{u}$ if, and only if:*

(i) u is a solution of (3.7),

(ii) $u \in [\underline{u}, \overline{u}]$ a.e. in Ω,

(iii) if $w \in [\underline{u}, \overline{u}]$ is any other solution of (3.7) then

$$w(x) \geq (or \ \leq) \ u(x)$$

a.e. $x \in \Omega$.

Our main goal in this section is to show the existence of a minimal solution and a maximal solution to (3.7) with the assumption that (3.7) has several subsolutions and supersolutions. The proof of the result is based on the techniques in [36] with some suitable modifications.

Theorem 3.23 *Assume that problem (3.7) has k subsolutions $\underline{u}_i$, $i = 1, \cdots, k$, and l supersolutions $\overline{u}_1, \cdots, \overline{u}_l$, $k, l \geq 1$, all of which are in $C^1(\Omega)$, such that*

$$\underline{u} := \max\{\underline{u}_i : i = 1, \cdots, k\} \leq \overline{u} := \min\{\overline{u}_j : j = 1, \cdots, l\} \qquad in\ \Omega.$$

Assume further that there exist a function $a_3 \in L^{\frac{p}{p-1}}(\Omega)$ and a constant $b_3 > 0$ such that

$$|f(x, s, \xi)| \leq a_3(x) + b_3|\xi|^p,$$

for all $s \in [\underline{u}_0(x), \overline{u}_0(x)]$, for a.e. $x \in \Omega$, where

$$\begin{aligned} \underline{u}_0 &:= \min\{\underline{u}_1, \cdots, \underline{u}_k\}, \\ \overline{u}_0 &:= \max\{\overline{u}_1, \cdots, \overline{u}_l\}. \end{aligned}$$

Then (3.7) has a minimal solution u_* and a maximal solution u^*, both of which are in $C^1(\Omega)$, with

$$\underline{u}(x) \leq u_*(x) \leq u^*(x) \leq \overline{u}(x), \qquad for\ a.e.\ x \in \Omega.$$

The proof of the existence of u_* will follow from several lemmas given below. The existence of u^* may be deduced in a similar fashion.

Let U denote the set of $C^1(\Omega)$ solutions of (3.7) between $\underline{u}$ and $\overline{u}$; that is,

$$U := \{u \in C^1(\Omega) : u \text{ is a solution of (3.7) such that } u \in [\underline{u}, \overline{u}]\}.$$

By Theorem 3.20, U is nonempty.

Lemma 3.24 *U is compact in $W_0^{1,p}(\Omega)$.*

Proof : Let

$$M := \|\underline{u}_0\|_{L^\infty(\Omega)} + \|\overline{u}_0\|_{L^\infty(\Omega)},$$

and let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence in U. Since u_n solves

$$\begin{cases} -\mathrm{div}A(x, \nabla u_n) &= f(x, u_n, \nabla u_n) \quad \text{in } \Omega, \\ u_n &= 0 \quad \text{on } \partial\Omega, \end{cases}$$

it also solves

$$\begin{cases} -\mathrm{div}A(x, \nabla u_n) &= \hat{f}(x, u_n, \nabla u_n) \quad \text{in } \Omega, \\ u_n &= 0 \quad \text{on } \partial\Omega, \end{cases}$$

for each $n \geq 1$, where

$$\hat{f}(x, s, \xi) := \begin{cases} f(x, \overline{u}(x), \xi) & s \geq \overline{u}(x), \\ f(x, s, \xi) & \underline{u}(x) \leq s \leq \overline{u}(x), \\ f(x, \underline{u}(x), \xi) & s \leq \underline{u}(x), \end{cases}$$

for a.e. $x \in \Omega$, all $s \in \mathbb{R}$ and all $\xi \in \mathbb{R}^N$.

Applying Proposition A.4 to the latter problem, we obtain the boundedness of $\{u_n\}_{n\in\mathbb{N}}$. Hence, this sequence contains a subsequence, still called $\{u_n\}_{n\in\mathbb{N}}$, weakly converging to some function u in $W_0^{1,p}(\Omega)$. Following the same arguments as used in the proof of Lemma 3.16, we get the strong convergence of $\{u_n\}_{n\in\mathbb{N}}$ to u. Finally, the fact $u \in U$ can be obtained by employing Theorem 1.Q in [54] and the dominated convergence theorem.

Lemma 3.25 *Every chain in U has a lower bound in U, with respect to the partial order $\leq$ (the usual partial order of real functions).*

Proof : Let S be a chain in U. Since U is uniformly bounded in $L^1(\Omega)$, so is S. Define

$$\delta = \inf_{u\in S}\left\{\int_\Omega u dx\right\} \geq \int_\Omega \underline{u} dx$$

and let $\{u_n\}_{n\in\mathbb{N}} \subset S$ be such that

$$\int_\Omega u_{n-1} dx \ \geq \ \int_\Omega u_n dx \geq \delta, \qquad n \geq 2, \tag{3.18}$$

$$\int_\Omega u_n dx \quad \to \quad \delta, \qquad \text{as } n \to \infty. \tag{3.19}$$

Since both u_{n-1} and u_n are in S, for each $n \geq 2$, either $u_{n-1} \leq u_n$ or $u_{n-1} \geq u_n$ holds. The former case (with proper inequality) cannot occur because of (3.18). Thus,

$$u_{n-1} \geq u_n, \qquad n \geq 2.$$

Since U is compact, we may assume that

$$u_n \to u, \qquad \text{in } W_0^{1,p}(\Omega)$$

and

$$u_n \to u, \qquad \text{in } L^1(\Omega),$$

for some $u \in S$. This, together with (3.19), implies

$$\int_\Omega u dx = \delta.$$

We now show that u is a lower bound of S. Let v be an arbitrary element of U. Consider two cases. If

$$\int_\Omega v dx = \delta,$$

then

$$\int_\Omega v dx \leq \int_\Omega u_n dx,$$

and, therefore,

$$v \leq u_n,$$

for all $n \in \mathbb{N}$ because S is a chain. Thus, $v \leq u$. This and the fact that

$$\int_\Omega v dx = \delta = \int_\Omega u dx$$

show $u = v$ in Ω. If

$$\int_\Omega v dx > \delta,$$

then there exists $n^* \in \mathbb{N}$ such that

$$\int_\Omega v dx \geq \int_\Omega u_n dx,$$

and hence,

$$v \geq u_n$$

for all $n \geq n^*$. This shows $v \geq u$ by letting n, in the inequality above, tend to ∞.

Using Zorn's Lemma, we obtain a minimal element u_* in U with respect to the partial order $\leq$. We now show that u_* is the minimal element of U. If u is an element of U such that $u \not\geq u_*$, it and u_* may be considered as two supersolutions of (3.7). Theorem 3.20 may be applied to find $u' \in U$ with

$$\underline{u} \leq u' \leq \min\{u_*, u\} \leq u_* \leq \overline{u}.$$

The minimality of u_* in U shows that $u' = u_*$ and that u_* is the minimal solution of (3.7) in $[\underline{u}, \overline{u}]$.

3.5 A sub-supersolution theorem for singular problems without convection terms

Assume that

$$f : \Omega \times (0, \infty) \to \mathbb{R}$$

is a Carathéodory function satisfying some growth condition given later. The aim of this section is to establish a sub-supersolution theorem for the following $p-$Laplace problem

$$\begin{cases} -\Delta_p u &= f(x,u) \quad \text{in } \Omega, \\ \quad u &> 0 \quad \text{in } \Omega, \\ \quad u &= 0 \quad \text{on } \partial\Omega. \end{cases} \tag{3.20}$$

We only study the $p-$Laplace equations here for simplicity and the obtained result is still true when $-\Delta_p$ in (3.20) is replaced by the Leray-Lions operator A in the previous sections. Finally, problem (3.20) is said to be singular on $\partial\Omega$ because $f(\cdot, 0)$ is undefined while its solution u satisfies the Dirichlet boundary condition that $u = 0$ on $\partial\Omega$.

We have the following theorem.

Theorem 3.26 *Assume that problem (3.20) has a subsolution $\underline{u}$ and a supersolution $\overline{u}$, in the sense of distributions, both of which belong to $L^\infty_{loc}(\Omega)$, such that*

$$0 < \underline{u}(x) \leq \overline{u}(x), \quad \textit{for a.e. } x \in \Omega.$$

Assume further, there exists a function $c \in L^\infty_{loc}(\Omega)$ such that,

$$|f(x,s)| \leq c(x), \quad \textit{for a.e. } x \in \Omega, \ \forall s \in [\underline{u}(x), \overline{u}(x)]. \tag{3.21}$$

Then problem (3.20) has a solution u in the sense of distributions and u satisfies

$$\underline{u} \leq u \leq \overline{u}, \qquad \textit{a.e. in } \Omega. \tag{3.22}$$

Proof : Let $\{\Omega_n\}_{n\in\mathbb{N}}$ be a sequence of smooth subdomains of Ω such that

$$\overline{\Omega}_n \subset \Omega_{n+1},\ n = 1, 2, \cdots,\ \cup_{n\in\mathbb{N}} \Omega_n = \Omega.$$

We proceed in the proof by establishing some auxiliary results.

Lemma 3.27 *There exists a sequence* $\{v_n\}_{n\in\mathbb{N}} \subset W_0^{1,p}(\Omega) \cap C(\overline{\Omega})$ *such that:*
(i) $0 \le v_1(x) \le v_2(x) \le \cdots \le \overline{u}(x) - \underline{u}(x)$, *for a.e.* $x \in \Omega$
and
(ii) for each $n \in \mathbb{N}$, *the restriction of* v_n *to* Ω_n *is a weak solution of*

$$\begin{cases} -\Delta_p(v_n + \underline{u}) &= f(x, v_n + \underline{u}) \quad \textit{in } \Omega_n, \\ v_n &= 0 \quad \textit{on } \partial\Omega_n. \end{cases}$$

Proof : Fix $n \in \mathbb{N}$. Note that $\underline{v} := 0$ and $\overline{v} := \overline{u} - \underline{u} \ge 0$ are, respectively, a subsolution and a supersolution (in the classical sense, see [35, 37, 38]) of

$$\begin{cases} -\Delta_p(v_n + \underline{u}) &= f(x, v_n + \underline{u}) \quad \text{in } \Omega_n, \\ v_n &= 0 \quad \text{on } \partial\Omega_n. \end{cases} \tag{3.23}$$

Since, for a.e. $x \in \Omega_n$, all $s \in (\underline{v}(x), \overline{v}(x))$,

$$|f(x, s + \underline{u})| \le c(x),$$

we may apply Remark 1.5 in [38] to find a minimal solution v_n, with respect to the pair $(\underline{v}, \overline{v})$, of problem (3.23) satisfying

$$\underline{v}(x) \le v_n(x) \le \overline{v}(x), \quad \text{for a.e. } x \in \Omega_n.$$

This means any other solution v_n' of (3.23), such that

$$\underline{v}(x) \le v_n'(x) \le \overline{v}(x), \quad \text{for a.e. } x \in \Omega_n,$$

must satisfy

$$v_n(x) \le v_n'(x), \quad \text{for a.e. } x \in \Omega_n.$$

Since $\overline{v} \in L^\infty(\Omega_n)$, so is v_n. Since v_n is essentially bounded, it follows from Corollary 1.5, [40], that v_n is Hölder continuous. We may therefore consider v_n as a function in $W_0^{1,p}(\Omega) \cap C(\overline{\Omega})$ by defining $v_n = 0$ in $\Omega \setminus \Omega_n$.

Next, we show

$$v_n(x) \le v_{n+1}(x),\ x \in \Omega,\ n = 1, 2, \cdots.$$

This inequality is clearly true when $x \in \Omega \setminus \Omega_n$. Assume then that there exists $n \in \mathbb{N}$ such that the Lebesgue measure of the set

$$\{y \in \Omega_n : v_n(y) > v_{n+1}(y)\}$$

is positive. We note that

$$v_{n+1}\,|_{\partial\Omega_n} \ge 0\ ,$$

and

$$\int_{\Omega_n} |\nabla(v_{n+1} + \underline{u})|^{p-2}\nabla(v_{n+1} + \underline{u}) \cdot \nabla\varphi dx = \int_{\Omega} f(x, v_{n+1} + \underline{u})\varphi dx,$$

for all $\varphi \in W_0^{1,p}(\Omega_n)$. Hence, v_{n+1} is a supersolution to (3.23) in the classical sense. We may apply Remark 1.5 in [38] again to find a solution w_n satisfying

$$0 \le w_n(x) \le \min\{v_n(x), v_{n+1}(x)\}\ \text{ a.e. }\ x \in \Omega_n.$$

Consequently,

$$w_n(x) < v_n(x),\ x \in \{y \in \Omega_n : v_n(y) > v_{n+1}(y)\}.$$

This, on the other hand, may not happen, because v_n is the minimal solution of (3.23).

Let u_n denote $v_n + \underline{u}$ for all $n \in \mathbb{N}$. The monotonicity of the sequence $\{v_n\}$ shows that $\{u_n\}$ converges to a function u at every point in $\overline{\Omega}$. We need to show that u is a solution of (3.20) in the sense of distributions.

Lemma 3.28 *For all domains $U \subset \Omega$, there exists a subsequence $\{u_{n_k}\} \subset \{u_n\}$ such that*

$$u_{n_k} \to u \ \textit{in}\ W^{1,p}(U).$$

Proof : Let $\varphi \in C_0^\infty(\Omega)$ be such that $0 \le \varphi \le 1$ in Ω and $\varphi = 1$ in U. Let K denote the support of φ. Without loss of generality, assume that $K \subset \Omega_n$ for all $n \in \mathbb{N}$. Since v_n is a solution of (3.23), applying Hölder's inequality and the product rule of differentiation, we obtain for $n = 1, 2, \cdots$,

$$\frac{1}{2^p}\int_K |\nabla(\varphi u_n)|^p dx \le C_1 + C_2\Big(\int_K |\nabla(\varphi u_n)|^p dx + C_3\Big)^{\frac{p-1}{p}},$$

where

$$C_1 \ \ = \ \ \int_K |\overline{u}\nabla(\varphi)|^p dx + \int_K c\varphi^p\overline{u}dx,$$

$$\begin{aligned} C_2 &= 2^p(\int_K |p\overline{u}\nabla\varphi|^p dx)^{\frac{1}{p}}, \\ C_3 &= \int_K |\overline{u}\nabla\varphi|^p dx. \end{aligned}$$

Therefore, $\{\varphi u_n\}$ is bounded in $W_0^{1,p}(K)$ and hence $\{u_n\}$ is bounded in $W^{1,p}(U)$. This implies that there exists a subsequence $\{u_{n_k}\} \subset \{u_n\}$ such that

$$u_{n_k} \rightharpoonup u \text{ in } W^{1,p}(U),$$

because $\{u_n\}$ converges to u pointwise in Ω. The process above may be applied again to find a subsequence of $\{u_{n_k}\}$, still called $\{u_{n_k}\}$, such that

$$u_{n_k} \rightharpoonup u \text{ in } W^{1,p}(K).$$

Next, we show

$$u_{n_k} \to u \text{ in } W^{1,p}(U).$$

It is sufficient to show that

$$|\nabla u_{n_k}| \to |\nabla u| \text{ in } L^p(U),$$

because it follows then from Lebesgue's convergence theorem, that

$$u_n \to u \text{ in } L^p(U).$$

Since v_{n_k} is a solution of

$$\begin{cases} -\Delta_p(v_{n_k} + \underline{u}) = f(x, v_{n_k} + \underline{u}) & \text{in } \Omega_{n_k}, \\ v_{n_k} = 0 & \text{on } \partial\Omega_{n_k}, \end{cases}$$

we have

$$\int_K |\nabla u_{n_k}|^{p-2}\nabla u_{n_k} \cdot \nabla(\varphi(u_{n_k} - u))dx = \int_K f(x, u_{n_k})\varphi(u_{n_k} - u)dx.$$

It follows from Lebesgue's convergence theorem and condition (3.21), that

$$\lim_{k\to\infty} \int_K f(x, u_{n_k})\varphi(u_{n_k} - u)dx = 0.$$

Thus,

$$\lim_{k\to\infty} \int_K |\nabla u_{n_k}|^{p-2}\nabla u_{n_k} \cdot \nabla(\varphi(u_{n_k} - u))dx = 0.$$

On the other hand, applying Hölder's inequality, we obtain

$$\left| \int_K (u_{n_k} - u)|\nabla u_{n_k}|^{p-2}\nabla u_{n_k} \cdot \nabla\varphi dx \right| \le C_4 \|u_{n_k} - u\|_{L^p(K)},$$

where

$$C_4 = \|\nabla\varphi\|_{L^\infty(K)} \sup_{k\in\mathbb{N}} \left\{ \|\nabla u_{n_k}\|^{p-1}_{L^p(K)} \right\}.$$

This, together with Lebesgue's convergence theorem, implies

$$\lim_{k\to\infty} \int_K \varphi |\nabla u_{n_k}|^{p-2} \nabla u_{n_k} \cdot \nabla(u_{n_k} - u) dx = 0.$$

Hence,

$$\lim_{k\to\infty} \int_K \varphi(|\nabla u_{n_k}|^{p-2} \nabla u_{n_k} - |\nabla u|^{p-2}\nabla u) \cdot \nabla(u_{n_k} - u) dx = 0.$$

Since the integrand is nonnegative and $\varphi = 1$ in U,

$$\lim_{k\to\infty} \int_U (|\nabla u_{n_k}|^{p-2} \nabla u_{n_k} - |\nabla u|^{p-2}\nabla u) \cdot \nabla(u_{n_k} - u) dx = 0.$$

It follows that

$$u_{n_k} \to u$$

in $W^{1,p}(U)$.

Let $\xi \in C_0^\infty(\Omega)$ and

$$V = \{x \in \Omega : \xi(x) \neq 0\}.$$

Since, for $n \gg 1$, $V \subset \Omega_n$, we have

$$\int_\Omega |\nabla u_n|^{p-2} \nabla u_n \cdot \nabla\xi dx = \int_\Omega f(x, u_n)\xi dx.$$

By Lemma 3.28, we may assume that $\{u_n\}$ converges to u in $W^{1,p}(V)$. Letting $n \to \infty$, we obtain the assertion of Theorem 3.26.

Remark 3.29 If both $\underline{u}$ and $\overline{u}$ are in $C(\overline{\Omega})$ and their value on $\partial\Omega$ is identically zero, then inequality (3.22) holds for all $x \in \Omega$ and u, therefore, satisfies the boundary condition

$$u = 0 \qquad \text{on } \partial\Omega.$$

Remark 3.30 One of the main points to prove Theorem 3.26 is the use of Remark 1.5 in [38] to construct the sequence $\{v_n\}_{n\in\mathbb{N}}$ of minimal solutions of (3.23). According to this remark, the existence of this sequence is still guaranteed if $\underline{u}$ is the maximum of a finite number of subsolutions, $\overline{u}$ is the minimum of several supersolutions to (3.20) and condition (3.21) holds for all s between the minimum of all subsolutions and the maximum of all supersolutions. The conclusion of Theorem 3.26 is true with these hypotheses because we can prove that $\{v_n + \underline{u}\}_{n\in\mathbb{N}}$ converges to the desired solution in $W^p(K)$ for all compact sets $K \subset \Omega$ by the same arguments as above.

3.6 Sub-supersolution theorems for singular equations with convection terms

We consider in this section a particular form of (3.1) when the domain of f is $\Omega\times(0,\infty)\times\mathbb{R}^N$ and $\mathcal{A}=-\Delta_p$. More precisely, we study the following problem

$$\begin{cases} -\Delta_p u &= f(x,u,\nabla u) \quad \text{in } \Omega, \\ \quad u &= 0 \qquad\qquad\quad \text{on } \partial\Omega, \end{cases} \tag{3.24}$$

Since $f(x,\cdot,\xi)$ is undefined at 0, $x\in\Omega$, $\xi\in\mathbb{R}^N$, (3.1) is singular. Also, problem (3.1) is said to involve convection terms because $\mathcal{F}u$ depends on ∇u, for all positive functions $u\in W_0^{1,p}(\Omega)$. We have studied sub-supersolution theorems for nonsingular equations with convection terms in Theorem 3.20 and Theorem 3.23. Since these theorems are not applicable to singular problems and we are interested in singular problems, we establish other sub-supersolution theorems for (3.24), one of which is stated below.

Theorem 3.31 *Let f be a Carathéodory function defined on $\Omega\times(0,\infty)\times\mathbb{R}^N$. Assume that problem (3.24) has k subsolutions $\underline{u}_1,\cdots,\underline{u}_k$ and l supersolutions $\overline{u}_1,\cdots,\overline{u}_l$, in the sense of distributions, all belonging to $C^1(\Omega)\cap L^\infty(\Omega)$, with*

$$\underline{u}:=\max\{\underline{u}_1,\cdots,\underline{u}_k\}\le\overline{u}:=\min\{\overline{u}_1,\cdots,\overline{u}_l\},$$

and that there exist a function $a_3\in L_{loc}^{\frac{p}{p-1}}(\Omega)$ and a constant $b_3>0$ such that

$$|f(x,s,\xi)|\le a_3(x)+b_3|\xi|^p, \tag{3.25}$$

for a.e. $x\in\Omega$, all $s\in[\underline{u}_0(x),\overline{u}_0(x)]$ where

$$\begin{aligned}\underline{u}_0 &:= \min\{\underline{u}_1,\cdots,\underline{u}_k\},\\ \overline{u}_0 &:= \max\{\overline{u}_1,\cdots,\overline{u}_l\}.\end{aligned}$$

Then, the first equation of (3.24) has a solution $u\in C^1(\Omega)\cap L^\infty(\Omega)$, in the sense of distributions, with $u\in[\underline{u},\overline{u}]$, i.e., for all $\phi\in C_0^\infty(\Omega)$,

$$\int_\Omega|\nabla u|^{p-2}\nabla u\cdot\nabla\phi dx=\int_\Omega f(x,u,\nabla u)\phi dx. \tag{3.26}$$

Moreover, if $\underline{u}$ and $\overline{u}$ are both identically 0 on $\partial\Omega$ then so is u and it solves (3.24).

Note that (3.25) is more general than (3.8) in the sense that a_3 is allowed to be in $L_{loc}^{\frac{p}{p-1}}(\Omega)$, rather than $L^{\frac{p}{p-1}}(\Omega)$. This will explain (see the next section) why Theorem 3.31 is applicable to singular problems.

Remark 3.32 Theorem 3.31 is similar to Theorem 1 in [2]. The main difference between these two theorems concerns the types of solution. Theorem 1 in [2] is concerned with classical solutions when $p = 2$ while Theorem 3.31 provides solutions in the sense of distributions.

Remark 3.33 The smoothness of the test function ϕ in (3.26) is not important. If u is a solution of (3.24) in the sense of distributions, then (3.26) is true for all $\phi \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ with compact support.

Remark 3.34 Theorem 3.31 is, of course, also valid, if the $p-$Laplace operator is replaced by a more general elliptic operator satisfying the conditions used earlier.

The proof of Theorem 3.31 is based on several lemmas given below.

Let $\{\Omega_n\}_{n\in\mathbb{N}}$ be a sequence of subdomains of Ω such that

$$\Omega = \bigcup_{n\in\mathbb{N}} \Omega_n$$

and

$$\Omega_1 \subset \overline{\Omega_1} \subset \Omega_2 \subset \overline{\Omega_2} \subset \cdots.$$

For each $n \in \mathbb{N}$, $i \in \{1, \cdots, k\}$ and $j \in \{1, \cdots, l\}$, $\underline{u}_i - \underline{u}$ and $\overline{u}_j - \underline{u}$ are subsolutions and supersolutions, respectively, in the sense of Definition 3.3 and Definition 3.4, of

$$\left\{ \begin{array}{rcll} -\Delta_p(v_n + \underline{u}) & = & f(x, v_n + \underline{u}, \nabla(v_n + \underline{u})) & \text{in } \Omega_n, \\ v_n & = & 0 & \text{on } \partial\Omega_n. \end{array} \right. \tag{3.27}$$

Noting that

$$0 = \max\{\underline{u}_1 - \underline{u}, \cdots, \underline{u}_k - \underline{u}\},$$

and

$$\overline{u} - \underline{u} = \min\{\overline{u}_1 - \underline{u}, \cdots, \overline{u}_l - \underline{u}\},$$

we may employ Theorem 3.23 to find a minimal solution $v_n \in [0, \overline{u} - \underline{u}]$ of (3.27). Define $v_n = 0$ on $\Omega \setminus \Omega_n$.

Lemma 3.35 *The sequence $\{v_n\}_{n\in\mathbb{N}}$ is increasing.*

Proof : Fix $n \geq 2$. Since 0 is the maximum of $\underline{u}_1, \cdots, \underline{u}_k$ and $v_n|_{\Omega_{n-1}}$ and v_{n-1} are two supersolutions of (3.27), we can find a solution w of (3.27) such that

$$0 \leq w \leq \min\{v_n|_{\Omega_{n-1}}, v_{n-1}\} \leq \overline{u} - \underline{u},$$

by Theorem 3.20. Since v_{n-1} is the minimal solution of (3.27) between the pair $[0, \overline{u} - \underline{u}]$,

$$v_{n-1} \leq w \leq \min\{v_n|_{\Omega_{n-1}}, v_{n-1}\} \leq v_n|_{\Omega_{n-1}},$$

which proves the lemma.

For all $n \in \mathbb{N}$, define $u_n = v_n + \underline{u}$. Since $\{v_n\}_{n\in\mathbb{N}}$ is increasing, so is $\{u_n\}_{n\in\mathbb{N}}$. Since $u_n \in [\underline{u}, \overline{u}]$ for all $n \in \mathbb{N}$, we can find a function $u \in [\underline{u}, \overline{u}]$ such that

$$\lim_{n\to\infty} u_n = u,$$

for a.e. $x \in \Omega$. The following lemma shows $u \in W^{1,p}_{loc}(\Omega)$ is the desired solution.

Lemma 3.36 *Let $K \subset \Omega$ be the closure of an open set. Then the sequence $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $W^{1,p}(K)$ for all such subsets.*

Proof : Since $\{u_n\}_{n\in\mathbb{N}}$ is uniformly bounded in $L^\infty(\Omega)$, it is sufficient to show that $\{|\nabla u_n|\}_{n\in\mathbb{N}}$ is bounded in $L^p(K)$.

Let $\varphi \geq 0$ be in $C_0^\infty(\Omega)$ such that $\varphi = 1$ in K. Define, for a positive number t, to be chosen later,

$$w_t = e^{tu_n^2}\varphi^p u_n,$$

whose gradient is

$$\nabla w_t = e^{tu_n^2}\varphi^p(2tu_n^2 + 1)\nabla u_n + pe^{tu_n^2}\varphi^{p-1}u_n\nabla\varphi.$$

Assume, without loss of generality, for all $n \geq 1$, that the support of φ is contained in Ω_n and hence $w_t \in W_0^{1,p}(\Omega_n)$. Now, using condition (3.25) and w_t as the test function in (3.27), we obtain

$$\begin{aligned}
&\int_\Omega e^{tu_n^2}\varphi^p(2tu_n^2 + 1)|\nabla u_n|^p dx + p\int_\Omega e^{tu_n^2}\varphi^{p-1}u_n|\nabla u_n|^{p-2}\nabla u_n\nabla\varphi dx \\
&\quad = \int_\Omega f(x, u_n, \nabla u_n)e^{tu_n^2}\varphi^p u_n dx \\
&\quad \leq \int_\Omega |a_3|e^{tu_n^2}\varphi^p|u_n|dx + b_3\int_\Omega |\nabla u_n|^p e^{tu_n^2}\varphi^p|u_n|dx
\end{aligned}$$

$$\leq \int_\Omega |a_3| e^{tu_n^2} \varphi^p |u_n| dx + b_3 \int_\Omega |\nabla u_n|^p e^{tu_n^2} \varphi^p \left(\frac{u_n^2}{2\epsilon} + \frac{\epsilon}{2} \right) dx,$$

for all $\epsilon > 0$. Since $\{u_n\}_{n\in\mathbb{N}}$ is uniformly bounded in $L^\infty(\Omega)$, $a_3 \in L_{loc}^{\frac{p}{p-1}}(\Omega)$ and $\varphi \in C_0^\infty(\Omega)$, we can find two numbers $b_{t,\varphi}$ and $c_{t,\varphi}$ such that

$$pe^{tu_n^2} |u_n \nabla\varphi| \leq b_{t,\varphi},$$

in Ω, and

$$\int_\Omega |a_3| e^{tu_n^2} \varphi^p |u_n| dx \leq c_{t,\varphi}.$$

Moreover, we can choose

$$t = \frac{b_3}{4\epsilon}$$

so that

$$\left(1 - \frac{\epsilon b_3}{2}\right) \int_\Omega e^{tu_n^2} \varphi^p |\nabla u_n|^p dx \quad \leq \quad c_{t,\varphi} + b_{t,\varphi} \left(\int_\Omega \varphi^p |\nabla u_n|^p dx \right)^{\frac{p-1}{p}}.$$

We now choose $\epsilon = \frac{1}{b_3}$ and get

$$\frac{1}{2} \int_\Omega \varphi^p |\nabla u_n|^p dx \leq c_{t,\varphi} + b_{t,\varphi} \left(\int_\Omega \varphi^p |\nabla u_n|^p dx \right)^{\frac{p-1}{p}}.$$

Here, we have used $e^{tu_n^2} \geq 1$. So, $\int_\Omega \varphi^p |\nabla u_n|^p dx$ is bounded uniformly in n. Therefore, $\{|\nabla u_n|\}_{n\in\mathbb{N}}$ is bounded in $L^p(K)$ because $\varphi = 1$ in K.

We now show that $u \in W_{loc}^{1,p}(\Omega)$. In fact, let Ω' be such that its closure, $\overline{\Omega'}$, is contained in Ω. Lemma 3.36 and the reflexivity of $W^{1,p}(\Omega')$ help us to find a weakly convergent subsequence of $\{u_n\}_{n\in\mathbb{N}}$ in $W^{1,p}(\Omega')$. The weak limit of such a subsequence must be u because $u_n \to u$ a.e. in Ω. Hence, $u \in W^{1,p}(\Omega')$.

Let $\phi \in C_0^\infty(\Omega)$ and let K denote the support of ϕ. Applying the arguments in the previous paragraph when Ω' is the interior of K, we have

$$u_n \rightharpoonup u \qquad \text{in } W^{1,p}(\Omega').$$

The following lemma can be proved using the same arguments used in the proof of Lemma 3.21.

Lemma 3.37 *$u_n \to u$ in $W^{1,p}(K)$.*

Recall that K is the support of ϕ. Letting $n \to \infty$ in the following equation

$$\int_K |\nabla u_n|^{p-2} \nabla u_n \cdot \nabla\phi dx = \int_K f(x, u_n, \nabla u_n)\phi dx,$$

implies that u solves (3.24) in the sense of distribution.

Fix $x \in \Omega$ and B is a ball containing x such that $\overline{B}$ is contained in Ω. It follows from Theorem 1.7 in [40] and the fact $u \in L^{\infty}(\Omega)$ that $u \in C^1(B)$. This completes the proof of Theorem 3.31.

Theorem 3.38 *If all assumptions of Theorem 3.31 hold, then (3.24) has a minimal and a maximal solution with respect to the pair* $(\underline{u}, \overline{u})$.

Proof : Let

$$U := \{u \in [\underline{u}, \overline{u}] : u \text{ solves (3.24) in the sense of distributions}\}$$

be the set of solutions of (3.24) lying between $\underline{u}$ and $\overline{u}$. Because of Theorem 3.31, U is nonempty. We may now employ the arguments used in Section 3.4 to show the existence of a minimal solution of (3.24) between $\underline{u}$ and $\overline{u}$.

Let S be a chain in U, with respect to the partial order $\leq$ and define δ as

$$\delta = \inf_{u \in S} \int_{\Omega} u dx \geq \int_{\Omega} \underline{u} dx > -\infty.$$

By definition of the infimum, it is possible to find a sequence $\{u_n\}_{n \in \mathbb{N}} \subset S$ such that

$$\int_{\Omega} u_{n+1} dx \leq \int_{\Omega} u_n dx$$

for all $n \geq 1$ and

$$\lim_{n \to \infty} \int_{\Omega} u_n dx = \delta.$$

Since S is a chain, $u_{n+1} \leq u_n$. Thus, $\{u_n\}_{n \in \mathbb{N}}$ converges to a function $u_* \in [\underline{u}, \overline{u}]$ a.e. in Ω. Employing the arguments in Lemma 3.36 and Lemma 3.37, we may show that $\{u_n\}_{n \in \mathbb{N}}$ is bounded and strongly converges to u in $W^{1,p}(K)$, for any compact subset K of Ω. Hence, $u \in U$. On the other hand, for all $w \in S$, if

$$\int_{\Omega} w dx = \delta$$

then

$$\int_{\Omega} w dx \leq \int_{\Omega} u_n dx,$$

for all $n \geq 1$. Since S is a chain, $w \leq u_n$ for all $n \geq 1$. Letting $n \to \infty$, we have $w \leq u$. This and the fact that

$$\int_{\Omega} w dx = \delta = \int_{\Omega} u dx$$

imply

$$w(x) = u(x)$$

a.e. in Ω. If

$$\int_\Omega w dx > \delta,$$

then it is easy to show $w \geq u$ a.e. in Ω. Both cases show that u is a lower bound of S. In other words, any chain in U has a lower bound.

Using Zorn's Lemma, we obtain a minimal element u_* in U with respect to the partial order $\leq$. We now show that u_* is the minimal element of U. If u is an element of U such that $u \not\geq u_*$, it and u_* belong to $C^1(\Omega)$ (by the arguments used in the last part of the proof of Theorem 3.31) and thus may be considered as two supersolutions of (3.24), in the sense of distributions. Theorem 3.31 may be applied to find $u' \in U$ with

$$\underline{u} \leq u' \leq \min\{u_*, u\} \leq u_* \leq \overline{u}.$$

The minimality of u_* in U shows that $u' = u_*$ and that u_* is the minimal solution of (3.24) in $[\underline{u}, \overline{u}]$.

The existence of a maximal solution to (3.24) may be established similarly.

3.7 Concluding remarks

We have established in this chapter sub-supersolution theorems for nonsingular and singular elliptic equations with or without convection terms. Our sub-supersolution theorems for nonsingular problems involving convection terms are motivated by and are more general than those in [36, 37, 38]. To prove these results, we employ a Leray Schauder degree technique defined on the class (S_+) and some special test functions similar to those in [54]. On the other hand, to prove sub-supersolution theorems for singular problems, we approximate the problems considered by nonsingular ones, obtained by replacing Ω by proper subdomains. Although Theorem 3.31 is applicable to a generalization of (3.20), it is not more general than Theorem 3.26 because the latter theorem does not require any smoothness conditions on the given sub-supersolutions.

CHAPTER 4

NONLINEAR SINGULAR ELLIPTIC EQUATIONS

We study the existence of positive solutions to singular elliptic boundary value problems involving the $p-$Laplace operator by applying the sub-supersolution theorems obtained in Chapter 3. Because of this, constructing subsolutions and supersolutions of the equations considered is the main point in this chapter. Our assumptions on singular terms and convection terms are more relaxed than in some previous papers, even for the case $p = 2$, as we allow for nonmonotone singular terms, with blowup controlled by a power, and non-Hölder continuous coefficients of the convection terms. We also allow for a parameter-dependent term and study how its growth affects our existence result.

4.1 Singular problems without convection terms

4.1.1 Some previous results of singular problems

We are interested in the following singular elliptic problem

$$\begin{cases} -\Delta_p u &= ag(u) + \lambda h(u) & \text{in } \Omega, \\ u &> 0 & \text{in } \Omega, \\ u &= 0 & \text{on } \partial\Omega, \end{cases} \tag{4.1}$$

where λ is a nonnegative parameter;

$$a : \Omega \to [1, \infty)$$

is in $L^\infty(\Omega)$;

$$g : (0, \infty) \to \mathbb{R}$$

is continuous and satisfies

$$\lim_{s \to 0^+} g(s) = \infty,$$

and,

$$h : [0, \infty) \to \mathbb{R}$$

is continuous.

Lazer and McKenna [34] have proved that (4.1) has a unique classical solution when $\lambda = 0$, $g(s) = s^{-\gamma}, s \in (0, \infty)$, $\gamma > 0$, and Ω is in class $C^{2+\beta}$, $\beta > 0$. Lair and Shaker [33] and Zhang and Cheng [58] have obtained the results of Lazer and McKenna (in the case $0 < \gamma < 1$) deducing the existence of solutions of

$$\begin{cases} -\Delta u &= ag(u) \quad \text{in } \Omega, \\ u &> 0 \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{cases}$$

where g is nonincreasing and satisfies

$$\int_0^1 g(s) ds < \infty.$$

Although Ω in [33] is either a bounded domain or the whole space $\mathbb{R}^N$, (while Ω in [58] is bounded) and the conditions imposed on a in [33] are weaker than those in [58], the results of [58] cannot be deduced from those of [33]. An additional significant paper is the paper by Crandall, Rabinowitz and Tartar [12], where the existence of solutions to the more general problem

$$Lu = g(x, u) \text{ in } \Omega, \qquad u = 0 \text{ on } \partial\Omega,$$

is studied, with L a linear second order elliptic operator which satisfies the maximum principle and g is positive and becomes singular as

$$u \to 0 \text{ uniformly in } x.$$

Their techniques are also based on the use of sub-supersolution theorems.

In the case that the problem depends on a parameter, several papers [11, 52, 53, 56] studied (4.1) when g and h are of particular forms. In particular, Coclite and Palmieri have proved in [11] that if $\alpha \geq 1$, then

$$\begin{cases} -\Delta u &= u^{-\gamma} + (\lambda u)^{\alpha} \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{cases} \tag{4.2}$$

has at least one solution when λ is small and (4.2) has no solution when λ is large. Using iteration techniques, the problem has also been studied by Sun and Wu [52] when $0 \leq \alpha < 1$, $0 < \gamma < N^{-1}$. Cîrstea, Gherghu and Rădulescu [9] have considered (4.1) for g nonincreasing, h nondecreasing and $p = 2$ and have proved (with some additional technical assumptions on g and h) that the problem

$$\begin{cases} -\Delta u &= ag(u) + \lambda h(u) \quad \text{in } \Omega, \\ u &> 0 \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega \end{cases} \tag{4.3}$$

has a unique solution u_λ for all $\lambda \geq 0$ and u_λ is increasing with respect to λ (i.e., $0 \leq \lambda_1 \leq \lambda_2$ implies $u_{\lambda_1} \leq u_{\lambda_2}$ in Ω), provided that

$$\lim_{s\to\infty} \frac{h(s)}{s} = 0 \text{ (see Theorem 1 in [9])},$$

and if

$$\lim_{s\to\infty} \frac{h(s)}{s} > 0,$$

then there exists $\lambda^* > 0$ such that (4.3) has a solution when $\lambda \in (0, \lambda^*)$ and has no solution when $\lambda \geq \lambda^*$ (see Theorem 2 in [9]). We also draw the reader's attention to the papers [19, 23] in which the existence and nonexistence of solutions to singular elliptic problems depending on two parameters were studied.

When $p \in (1, \infty)$, by using a sub-supersolution approach and a mountain pass theorem, Giacomoni, Schindler and Takáč [22] have proved that

$$\begin{cases} -\Delta_p u &= \lambda u^{-\delta} + u^q \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{cases}$$

where $\delta \in (0, 1)$, $q \in (p-1, p^*-1)$ (p^* is the critical Sobolev exponent defined by p), has multiple weak solutions (depending on the value of the parameter λ).

All of the papers mentioned above needed a monotonicity condition on the singular term g. Thus the question arises whether or not the existence of solutions for (4.1) is still true when the monotonicity property is removed. Hai, [24, 25], has given affirmative answers to this question in the case that Ω is an annulus, by establishing existence results for radial solutions which are solutions of associated ordinary differential equations.

We approach to solve (4.1) by using Theorem 3.26 and then finding a well-ordered pair of sub-supersolutions for the specific singular problem under consideration. With this method, we can remove not only the monotonicity condition but also some technical conditions on the singular terms in the papers above.

4.1.2 Hopf's Lemma

In this section, we shall recall Hopf's Lemma which is needed to prove some properties of eigenfunctions associated to the first eigenvalue λ_1 of $-\Delta_p$. Let $\phi \in C^1(\overline{\Omega})$ be a solution of

$$\begin{cases} -\Delta_p \phi &= \lambda_1 \phi^{p-1} \quad \text{in } \Omega, \\ \phi &> 0 \quad \text{in } \Omega, \\ \phi &= 0 \quad \text{on } \partial\Omega \end{cases} \tag{4.4}$$

(cf. [40], [41]). The following lemma is well-known when $p = 2$ and is a corollary of Lemma A.3 in [48].

Lemma 4.1 *For all $x \in \Omega$*

$$\frac{\partial \phi(x)}{\partial \nu} < 0,$$

where ν is the outward unit normal vector to $\partial\Omega$ at x.

Note that the maximum principle of Vázquez [55] is not applicable, since it requires $\Delta_p \phi \in L^2_{loc}(\Omega)$.

The following lemma gives a property of the eigenfunction in Lemma 4.1, which we will need to prove Remark 4.7.

Lemma 4.2 *Let $\varphi \in C^1(\overline{\Omega})$. Assume that for all $x \in \partial\Omega$,*

$$\frac{\partial \varphi(x)}{\partial \nu} < 0.$$

Then

$$\int_\Omega \varphi^r dx < \infty,$$

if, and only if $r > -1$.

Lazer and McKenna [34] have proved this lemma for the eigenfunction $\varphi = \phi$ when $p = 2$. The general case may be proved in a similar way. (Note that this result is a general result implied by the behavior of the function at $\partial\Omega$.)

4.1.3 The existence result and some remarks

In this section, we shall present the main result of this chapter, Theorem 4.3, and its proof. As mentioned, we shall employ arguments using one of the the sub-supersolution theorems proved earlier. Thus, the main point here is the construction of a well-ordered pair of sub-supersolutions of (4.1).

Theorem 4.3 *Assume g satisfies:*

$$\exists \gamma > 0,\ C > 0 \text{ such that } g(s) \leq C s^{-\gamma},\ \forall s \in (0, \infty). \tag{4.5}$$

Then:
(i) if $\limsup_{s \to 0^+} \frac{h(s)}{s^{p-1}} < \infty$, there exists $\tilde{\lambda} > 0$ such that for all $\lambda \in [0, \tilde{\lambda}]$, problem (4.1) has a solution,
(ii) if there exists $\alpha < p - 1$ such that

$$0 \leq h(s) \leq s^\alpha,\ \forall s \in [1, \infty),$$

then for all $\lambda \geq 0$, problem (4.1) has a solution.

Proof : For each $b > 0$, define the function Ψ_b on Ω as follows

$$\Psi_b = b\phi^t,$$

where $t \in (0,1)$ is such that

$$t(p-1+\gamma) \leq p,\ tp - t + \gamma t - p \leq 0. \tag{4.6}$$

Note that equalities in (4.6) can be satisfied when $\gamma > 1$. A direct calculation shows that ϕ is a weak solution of

$$-\Delta_p\left(b\phi^t(x)\right) = (bt)^{p-1}\phi^{tp-t-p}(x)\left[q(t,x) + \frac{\lambda_1\phi^p(x)}{2}\right],$$

or equivalently,

$$-\Delta_p\Psi_b(x) = (bt)^{p-1}\phi^{tp-t-p}(x)\left[q(t,x) + \frac{\lambda_1\phi^p(x)}{2}\right], \tag{4.7}$$

where

$$q(t,x) = (1-t)(p-1)|\nabla\phi(x)|^p + \frac{\lambda_1\phi^p(x)}{2}.$$

It follows from Lemma 4.1 that $\nabla\phi \neq 0$ on $\partial\Omega$. So, there exists $\beta > 0$, depending on t, such that $q(t,x) > \beta$, $x \in \overline{\Omega}$.

Lemma 4.4 *Assume that* $\limsup\limits_{s\to 0^+} \dfrac{h(s)}{s^{p-1}} < \infty$. *Then there exists* $\tilde{\lambda} > 0$, *such that for all* $\lambda \in [0,\tilde{\lambda}]$, *problem (4.1) has a supersolution* $\overline{u} \in L^\infty(\Omega)$.

Proof : When b is large, with the help of (4.5) and (4.6), we conclude that

$$\begin{aligned}
\beta(bt)^{p-1}\phi^{tp-t-p} - ag(\Psi_b) &\geq \beta(bt)^{p-1}\phi^{tp-t-p} - C\|a\|_{L^\infty(\Omega)}\Psi_b^{-\gamma} \\
&= \beta(bt)^{p-1}\phi^{tp-t-p} - C\|a\|_{L^\infty(\Omega)}(b\phi^t)^{-\gamma} \\
&\geq \phi^{-t\gamma}\left[\beta(bt)^{p-1}\phi^{tp-t-p+t\gamma} - \frac{C\|a\|_{L^\infty(\Omega)}}{b^\gamma}\right] \\
&\geq 0,
\end{aligned}$$

where the constant C in the above calculation is given by (4.5). Thus,

$$(bt)^{p-1}\phi^{tp-t-p}(x)q(t,x) - a(x)g(\Psi_b(x)) \geq 0,\ x \in \Omega. \tag{4.8}$$

Now, choose $\tilde{\lambda}$, small enough, so that

$$\tilde{\lambda}h(s) < \frac{\lambda_1 t^{p-1}}{2}s^{p-1},\ \forall s \in (0, \max_{x\in\overline{\Omega}}\Psi_b^t(x)].$$

For all $\lambda \in [0,\tilde{\lambda}]$,

$$\frac{\lambda_1}{2}(bt)^{p-1}\phi^{tp-t} - \lambda h(\Psi_b) \geq \frac{\lambda_1}{2}(bt)^{p-1}\phi^{t(p-1)} - \frac{\lambda_1 t^{p-1}}{2}\Psi_b^{p-1}$$

$$
\begin{aligned}
&= \frac{\lambda_1}{2}(bt)^{p-1}\phi^{t(p-1)} - \frac{\lambda_1(bt)^{p-1}}{2}\phi^{t(p-1)} \\
&= 0.
\end{aligned}
$$

This, (4.7) and (4.8), imply that $\overline{u} = \Psi_b$ is a supersolution of (4.1).

Lemma 4.5 *Assume that there exists $\alpha < p-1$ such that*

$$0 \leq h(s) \leq s^{\alpha}, \ \forall s \in [1, \infty).$$

Then for all $\lambda \geq 0$, (4.1) has a supersolution $\overline{u} \in L^{\infty}(\Omega)$.

Proof : We first choose b, large, such that

$$\frac{1}{2}\beta b^{p-1+\gamma} t^{p-1} \min_{x\in\overline{\Omega}}\{\phi^{tp-t-p+\gamma t}(x)\} \geq C\|a\|_{L^{\infty}(\Omega)} \tag{4.9}$$

and

$$\frac{1}{2}\beta(bt)^{p-1} \geq \lambda M \max_{x\in\overline{\Omega}}\{\phi(x)\}^{t+p-tp}, \tag{4.10}$$

where

$$M = \max_{s\in[0,\Lambda]} h(s), \ \Lambda = \max\left\{\left(\frac{2\lambda}{\lambda_1 t^{p-1}}\right)^{\frac{1}{p-1-\alpha}}, 1\right\}.$$

Define $\overline{u} := \Psi_b$. The choice of b in (4.9) implies

$$\frac{1}{2}\beta b^{p-1} t^{p-1}\phi^{tp-t-p} \geq ag(\overline{u}), \text{ in } \Omega. \tag{4.11}$$

Using (4.7), (4.10) and (4.11), we obtain

$$
\begin{aligned}
-\Delta_p \overline{u} &\geq ag(\overline{u}) + \frac{\beta(bt)^{p-1}\phi^{tp-t-p}}{2} + \frac{\lambda_1 t^{p-1}\overline{u}^{p-1}}{2} \\
&\geq ag(\overline{u}) + \lambda h(\overline{u}) + \frac{\lambda_1 t^{p-1}\overline{u}^{p-1}}{2} \\
&\geq ag(\overline{u}) + \lambda h(\overline{u}),
\end{aligned}
$$

on the set $\{x \in \Omega : 0 < \overline{u}(x) \leq \Lambda\}$. On the complementary set, $\overline{u} \geq 1$, and

$$
\begin{aligned}
\frac{\lambda_1 t^{p-1}\overline{u}^{p-1}}{2} &= \frac{\lambda_1 t^{p-1}\overline{u}^{p-1-\alpha}}{2}\overline{u}^{\alpha} \\
&\geq \frac{\lambda_1 t^{p-1}\Lambda^{p-1-\alpha}}{2}h(\overline{u}) \\
&\geq \lambda h(\overline{u}).
\end{aligned}
$$

Hence, by (4.7), (4.10) and (4.11)

$$
\begin{aligned}
-\Delta_p \overline{u} &\geq ag(\overline{u}) + \frac{\beta(bt)^{p-1}\phi^{tp-t-p}}{2} + \lambda h(\overline{u}) \\
&\geq ag(\overline{u}) + \lambda h(\overline{u}),
\end{aligned}
$$

whenever $\overline{u} \geq \Lambda$. So, $\overline{u}$ is a supersolution of (4.1).

Next, we find a subsolution for (4.1). Since

$$\lim_{\epsilon\to 0^+}\left(g(\epsilon\phi(x))+\lambda h(\epsilon\phi(x))\right)=\infty$$

and

$$\lim_{\epsilon\to 0^+}(\epsilon\phi(x))^{p-1}=0,$$

uniformly with respect to $x\in\Omega$, we can find $\epsilon>0$ and $M>0$ such that

$$\lambda_1(\epsilon\phi(x))^{p-1}<M<g(\epsilon\phi(x))+\lambda h(\epsilon\phi(x)),\ x\in\Omega.$$

Thus, for all $\varphi\in C_0^\infty(\Omega)$, $\varphi\geq 0$

$$\begin{aligned}\int_\Omega |\nabla(\epsilon\phi)|^{p-2}\nabla(\epsilon\phi)\cdot\nabla\varphi dx &= \int_\Omega \lambda_1(\epsilon\phi)^{p-1}\varphi dx\\ &\leq \int_\Omega \left(g(\epsilon\phi)+h(\epsilon\phi)\right)\varphi dx.\end{aligned}$$

It follows that $\underline{u}=\epsilon\phi$ is a subsolution of (4.1).

Since the supersolution $\overline{u}$, obtained in Lemma 4.4 and Lemma 4.5, is of the form

$$\overline{u}=b\phi^t,$$

for some $b>0$ and $0<t<1$, we can find ϵ small enough that

$$\underline{u}(x)\leq\overline{u}(x),\ x\in\Omega.$$

It follows from Theorem 3.26 and Remark 3.29 that there exists a solution u of (4.1) satisfying

$$\epsilon\phi(x)\leq u(x)\leq b\phi^t(x),\qquad x\in\Omega$$

for some $0<\epsilon\ll 1$, $b\gg 1$, and $0<t<1$.

Remark 4.6 It follows from Theorem 4.3 that there exist $b>0$ and $t\in(0,1)$ such that

$$0<u(x)<b\phi^t(x),\ x\in\Omega,$$

where u is a solution of (4.1) obtained by Theorem 4.3.

When $\gamma\geq\frac{2p-1}{p-1}$, we let $t=\frac{p}{p-1+\gamma}\in(0,1)$ so that the inequalities in condition (4.6) hold. In this case, under an additional condition on g, the solution u in Theorem 4.3 is not a weak solution of (4.1). This is illustrated by Example 1.1 and is shown by the following remark.

Remark 4.7 Assume in addition to (4.5), that g satisfies

$$g(s) \geq C^{-1}s^{-\gamma},\ \forall s > 0, \tag{4.12}$$

where C and γ are defined in (4.5). Then if $\gamma \geq \frac{2p-1}{p-1}$, the solution u obtained in Theorem 4.3 is not in $W_0^{1,p}(\Omega)$.

Proof : Let u be the solution obtained from Theorem 4.3. It follows from Remark 4.6 that there exists $b > 0$ such that

$$u(x) \leq b\phi^{\frac{p}{p-1+\gamma}}(x), \quad \text{for a.e. } \ x \in \Omega.$$

Thus

$$\int_\Omega au^{1-\gamma}dx \geq \int_\Omega a(b\phi^{\frac{p}{p-1+\gamma}})^{1-\gamma}dx = \infty, \tag{4.13}$$

which follows from Lemma 4.2.

Suppose, contrary to the assertion of the remark, that $u \in W_0^{1,p}(\Omega)$. Then, there exists a sequence $\{w_n\} \subset C_0^\infty(\Omega)$ such that

$$w_n \to u \text{ in } W_0^{1,p}(\Omega).$$

Define

$$w_n^+ := \max\{w_n, 0\} \in W_0^{1,p}(\Omega),\ n = 1, 2, \cdots.$$

Since $u \geq 0$,

$$w_n^+ \to u \text{ in } W_0^{1,p}(\Omega).$$

Without loss of generality, assume that w_n^+ converges to u almost everywhere in Ω. Using Fatou's Lemma and inequality (4.13), we obtain

$$\lim_{n\to\infty}\int_\Omega w_n^+au^{-\gamma}dx = \infty.$$

Since $w_n^+ \in W_0^{1,p}(\Omega)$ and condition (4.12) holds, we have

$$\begin{aligned}\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla w_n^+dx &= \int_\Omega (aw_n^+g(u) + \lambda h(u)w_n^+)dx \\ &\geq \int_\Omega (C^{-1}aw_n^+u^{-\gamma} + \lambda h(u)w_n^+)dx.\end{aligned}$$

Hence

$$\int_\Omega |\nabla u|^pdx = \lim_{n\to\infty}\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla w_n^+dx = \infty,$$

which contradicts the assumption that $u \in W_0^{1,p}(\Omega)$.

When $p = 2$, we may use regularity techniques from [34] to show that the solution obtained in Theorem 3.26 is a classical solution, provided that the function f is Lipschitz continuous and $\underline{u} \in C^2(\Omega) \cap C(\overline{\Omega})$. Thus, if a, g and h are Lipschitz continuous, then the solution u obtained in Theorem 4.3 is a classical solution.

4.2 Singular equations involving convection terms

4.2.1 Equations without parameter dependence

As in Subsection 4.1.2, let λ_1 be the first (principal) eigenvalue of $-\Delta_p$ and let ϕ denote a positive eigenfunction of $-\Delta_p$ associated to λ_1; i.e., ϕ solves

$$\begin{cases} -\Delta_p \phi &= \lambda_1 |\phi|^{p-2}\phi \quad \text{in } \Omega, \\ \phi &> 0 \quad \text{in } \Omega, \\ \phi &= 0 \quad \text{on } \partial\Omega. \end{cases}$$

Similar to Subsection 4.1.3, our main purpose in this subsection is to employ ϕ to construct a well-ordered pair of sub-supersolutions to

$$\begin{cases} -\Delta_p u &= g(x,u) + h(x, \nabla u) \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{cases} \tag{4.14}$$

and then seek a solution for this problem.

Here, g and h are two Carathéodory functions defined on $\Omega \times (0,\infty)$ and $\Omega \times \mathbb{R}^N$, respectively. We assume throughout this subsection that there exist $C_1 > 0$, $C_2 > 0$, and $\alpha > 0$ such that for a.e. $x \in \Omega$,

$$g(x,s) \le C_1 s^{-\alpha}, \qquad \forall s > 0,$$

$$g(x,s) > 1 \qquad \forall s \in (0,1),$$

and

$$|h(x,\xi)| \le C_2 |\xi|^p, \qquad \forall \xi \in \mathbb{R}^N.$$

Note that $g(x,\cdot)$ is allowed to have a singularity at 0^+ for a.e. $x \in \Omega$.

Lemma 4.8 *Problem (4.14) has a supersolution.*

Proof : Let $b > 1$ be so large that

$$tp - t - p + t\alpha < 0,$$

where

$$t = b^{-1-\frac{\alpha}{2(p-1)}}.$$

Define

$$\Psi_b = b\phi^t.$$

We have, as in (4.7),

$$\nabla \Psi_b = bt\phi^{t-1}\nabla\phi,$$

$$-\Delta_p \Psi_b \quad = \quad (bt)^{p-1}\phi^{tp-t-p}\left[(1-t)(p-1)|\nabla\phi|^p + \lambda_1\phi^p\right].$$

Thus,

$$\begin{aligned}
-\Delta_p \Psi_b &= (bt)^{p-1}\phi^{tp-t-p}\left[\frac{1}{2}(1-t)(p-1)|\nabla\phi|^p + \lambda_1\phi^p\right] \\
&\quad +\frac{1}{2}(1-t)(p-1)(bt)^{p-1}\phi^{tp-t-p}|\nabla\phi|^p \\
&= b^{\frac{\alpha}{2}}\phi^{tp-t-p+t\alpha}\left[\frac{1}{2}(1-t)(p-1)|\nabla\phi|^p + \lambda_1\phi^p\right]\Psi_b^{-\alpha} \\
&\quad +\frac{1}{2}(1-b^{-1-\frac{\alpha}{2(p-1)}})(p-1)b^{\frac{\alpha}{2(p-1)}}\phi^{-t}|\nabla\Psi_b|^p.
\end{aligned}$$

Since b may be chosen arbitrarily large, the last quantity above is dominated by

$$g(x,\Psi_b) + h(x,\nabla\Psi_b).$$

In other words, $\overline{u} = \Psi_b$ is a supersolution of (4.14) for b large.

Let $\epsilon > 0$ be chosen so small that

$$\epsilon^{p-1}\lambda_1\phi^{p-1} + C_2|\nabla(\epsilon\phi)|^p < 1.$$

Assuming also that

$$\epsilon\phi < 1,$$

we have

$$-\Delta_p(\epsilon\phi) - |h(x,\epsilon\phi)| < 1 < g(x,\epsilon\phi).$$

This gives us the following Lemma.

Lemma 4.9 *If $0 < \epsilon \ll 1$, then the function $\underline{u} = \epsilon\phi$ is a subsolution of (4.14).*

We are now in the position to prove an existence result, which is also our main theorem in this section.

Theorem 4.10 *Problem (4.14) has a minimal solution and a maximal solution with respect to the pair $(\epsilon\phi, b\phi^t)$, where $\epsilon > 0$ is sufficiently small, b is sufficiently large, ϕ is a first (principal) positive eigenfunction of $-\Delta_p$ and $t = b^{-1-\frac{\alpha}{2(p-1)}}$.*

Proof : It follows from Lemma 4.8 and Lemma 4.9 that

$$\overline{u} = b\phi^t$$

is a supersolution and

$$\underline{u} = \epsilon\phi$$

is a subsolution of (4.14). Since $t = b^{-1-\frac{\alpha}{2(p-1)}} < 1$, then if $\phi < 1$, we have $\phi \leq \phi^t$ and therefore, $\underline{u} \leq \overline{u}$. Otherwise, when $\phi \geq 1$, $b\phi^t > 1 > \epsilon\phi$. Both cases yield

$$\underline{u} \leq \overline{u}$$

in Ω. The theorem then follows by applying Theorem 3.38.

4.2.2 Singular problems involving parameter dependent terms

Let $g : (0, \infty) \to [0, \infty)$ be continuous and $h : \Omega \times [0, \infty) \to \mathbb{R}$ be a Carathéodory function. Note that g might be singular at 0^+. Motivated by [21, 44], we consider the following problem

$$\begin{cases} -\Delta_p u + k_1|\nabla u|^r &= k_2 g(u) + \lambda h(x,u) & \text{in } \Omega, \\ u &> 0 & \text{in } \Omega, \\ u &= 0 & \text{on } \partial\Omega, \end{cases} \tag{4.15}$$

where $k_1 \geq 0$ and $k_2 \geq 0$ are nonnegative $L^\infty(\Omega)$ functions, $r \in [0, p]$, and λ is a nonnegative parameter.

Theorem 4.11 *Assume there exist $C > 0$ and $\alpha > 0$ such that*

$$g(s) \leq Cs^{-\alpha} \qquad \forall s \in (0, \infty).$$

Then the following hold:

(i) If $\limsup_{s\to 0^+} \dfrac{h(x,s)}{s^{p-1}} < \infty$ uniformly in $x \in \Omega$, there exists $\tilde{\lambda} > 0$ such that for all $\lambda \in [0, \tilde{\lambda}]$, problem (4.15) has a solution.

(ii) If there exists $q < p - 1$ such that

$$0 \leq h(x,s) \leq s^q, \ \forall s \in [1, \infty),$$

uniformly in $x \in \Omega$, then for all $\lambda \geq 0$, problem (4.15) has a solution.

Proof : Employing Theorem 4.3, we can find a solution $\overline{u} \in W^{1,p}_{loc}(\Omega)$, in the sense of distributions, of

$$\begin{cases} -\Delta_p \overline{u} &= k_2 g(\overline{u}) + \lambda h(x, \overline{u}) \quad \text{in } \Omega, \\ \overline{u} &> 0 \quad \text{in } \Omega, \\ \overline{u} &= 0 \quad \text{on } \partial\Omega, \end{cases} \tag{4.16}$$

with

$$\overline{u} \geq \epsilon\phi,$$

for all $0 < \epsilon \ll 1$, where ϕ is a principal positive eigenfunction of $-\Delta_p$. Note that the existence of $\overline{u}$ depends on the growth of h, described in (i) and (ii). If

$$\limsup_{s \to 0^+} \frac{h(x,s)}{s^{p-1}} < \infty,$$

uniformly in $x \in \Omega$, then there exists $\tilde{\lambda}$ such that the existence of $\overline{u}$ is guaranteed when $\lambda \in [0, \tilde{\lambda}]$. If there exists $q < p - 1$ such that

$$0 \leq h(x,s) \leq s^q, \ \forall s \in [1, \infty),$$

uniformly in $x \in \Omega$, then for all $\lambda \geq 0$, (4.16) is solvable in the sense of distributions. Applying the regularity results as in the last part of the proof of Theorem 3.31, we have $\overline{u} \in C^1(\Omega)$. Obviously, $\overline{u}$ is a supersolution of (4.15). On the other hand, $\underline{u} = \epsilon\phi$ may be shown to be a subsolution of (4.15), as was done in Lemma 4.9. Thus, Theorem 3.31 may be used to prove the existence of a solution of (4.15).

4.2.3 Concluding remarks

Our approach to study a class of singular elliptic problems with or without convection terms is the use of sub-supersolution theorems. This approach allows for the removal of the monotonicity and other technical conditions required of the singular terms in [11, 12, 34, 33, 52, 53, 56, 58]. Moreover, the local Hölder continuity requirement on the convection terms needed in [1, 15, 20, 21] may be removed. To illustrate this, we introduce two results which have motivated our study of problems of the form (4.14).

Recently, Alves, Carrião and Faria [1] established the following existence result for singular elliptic equations.

Theorem 4.12 *Assume that:*
(a) $h : \Omega \times \mathbb{R} \to \mathbb{R}$ *and* $g : \Omega \times \mathbb{R}^N \to \mathbb{R}$ *are locally Hölder continuous.*

(b) There exist constants $b > 0$, $0 < r_i < 1$, $(i = 1, 2, 3)$ with $r_1 < r_2$, and positive continuous functions $a_i : \overline{\Omega} \to \mathbb{R}$, $(i = 1, 2, 3)$ such that

$$b|\mu|^{r_1} \leq h(x, \mu) \leq a_1(x) + a_2(x)|\mu|^{r_2} + a_3(x)|\mu|^{-r_3}, \qquad \forall (x, \mu) \in \Omega \times \mathbb{R}.$$

(c) There exist a constant $0 < r_4 < 1$, and continuous functions a_4 and a_5 such that

$$0 \leq g(x, \eta) \leq a_4(x) + a_5(x)|\eta|^{r_4}, \qquad \forall (x, \eta) \in \Omega \times \mathbb{R}^N.$$

Then

$$\begin{cases} -\Delta u &= h(x, u) + g(x, \nabla u) \quad \text{in } \Omega, \\ \quad u &= 0 \qquad\qquad\qquad\quad\; \text{on } \partial\Omega, \end{cases}$$

has a solution.

Instead of employing a Galerkin method as in [1], we may apply Theorem 4.10 to obtain Theorem 4.12. Moreover, with this approach, h, g, and a_i, $i = 1, \cdots, 5$, are not required to be Hölder continuous.

Also, Theorem 4.11 may be used to deduce the existence result of Ghergu and Rădulescu in [21], which is stated below.

Theorem 4.13 *Let $K < 0$ be in $C^{0,\gamma}(\overline{\Omega})$, $0 < \gamma < 1$, $f : \overline{\Omega} \times [0, \infty) \to [0, \infty)$ be a Hölder continuous function which is positive on $\overline{\Omega} \times (0, \infty)$ and $g \in C^{0,\gamma}(\overline{\Omega})$ is a nonnegative and nonincreasing function.*

Assume that:

(i) the mapping $(0, \infty) \ni s \mapsto \frac{f(x,s)}{s}$ is nonincreasing for all $x \in \overline{\Omega}$,

(ii) $\lim_{s \to 0^+} \frac{f(x,s)}{s} = \infty$ and $\lim_{s \to \infty} \frac{f(x,s)}{s} = 0$ uniformly for $x \in \overline{\Omega}$,

(iii) $\lim_{s \to 0^+} g(s) = \infty$.

Then, for $0 \leq a \leq 2$, the problem

$$\begin{cases} -\Delta u + K(x)g(u) + |\nabla u|^a &= f(x, u) \quad \text{in } \Omega, \\ \qquad\qquad\qquad\quad u &> 0 \qquad\quad\;\; \text{in } \Omega, \\ \qquad\qquad\qquad\quad u &= 0 \qquad\quad\;\; \text{on } \partial\Omega, \end{cases} \tag{4.17}$$

has a solution.

Note that the Hölder continuity of f and g may be removed. Also, conditions (i), (ii) and (iii) may be replaced by the requirement that there exist $0 < q < 1$ and $c > 0$ such that

$$|f(x, s)| \leq cs^q,$$

for a.e. $x \in \Omega$, all $s \geq 0$.

CHAPTER 5

NONLOCAL PROBLEMS MODELING SHEAR BANDINGS

In this chapter, we are interested in the case that the operator $\mathcal{F}$ in (1.1) involves nonlocal terms. Precisely, we study the following problem

$$\begin{cases} -\Delta_p u &= \dfrac{\lambda}{\left(\fint_\Omega g(u)dy\right)^r} f(u) \quad \text{in } \Omega, \\ u &= 0 \qquad\qquad\qquad\qquad \text{on } \partial\Omega, \end{cases} \tag{5.1}$$

where r is a nonnegative number, λ is a nonnegative parameter and f and g are two positive continuous functions with $g(s) > 1$ for all $s \in \mathbb{R}$. Problem (5.1) is a generalization of the problem modeling shear bandings

$$\begin{cases} -\Delta_p u &= \dfrac{\lambda e^u}{\left(\fint_\Omega e^u dy\right)^r} \quad \text{in } \Omega, \\ u &= 0 \qquad\qquad\quad \text{on } \partial\Omega, \end{cases} \tag{5.2}$$

which was introduced and studied by Bebernes and Talaga [4] and then by Miyasita [46] when $p = 2$ and Ω is radially symmetric. These two papers completely described the structure of solutions of (5.2). Moreover, when $r = 0$, (5.2) is known as the Liouville-Bratu-Gelfand problem and was studied, when Ω is a ball in $\mathbb{R}^N$, in [42] when $p = 2$ and in [29, 30] when $p > 1$. These works completely described the continuum of solutions to the Liouville-Bratu-Gelfand problem, depending on the dimension N. Also Jacobsen [28] studied this problem and obtained an unbounded continuum of solutions, when the $p-$Laplacian is replaced by the $k-$Hessian operators and Ω is not necessarily a ball. This paper also described the structure of the continuum when Ω is a ball for all k.

In this chapter, we also show the existence of an unbounded continuum of solutions and find $\Lambda > 0$ so that (5.1) has a solution for all $\lambda \in [0, \Lambda]$ for any smooth bounded domain Ω in $\mathbb{R}^N$.

5.1 The existence of an unbounded continuum

We begin this section by recalling a boundary regularity result for solutions of degenerate elliptic equations, which plays an important role in our arguments. This regularity result can be deduced from Theorem 1 in [39].

Lemma 5.1 *Let u be a bounded weak solution of the problem*

$$\begin{cases} -\Delta_p u &= h(x) \quad in\ \Omega, \\ u &= 0 \quad on\ \partial\Omega, \end{cases}$$

If h belongs to $L^\infty(\Omega)$ with $\|h\|_{L^\infty(\Omega)} \le H$ then there exist $\alpha = \alpha(H) \in (0,1)$ and $C = C(H) > 0$ such that $u \in C^{1,\alpha}(\overline{\Omega})$ and

$$\|u\|_{C^{1,\alpha}(\overline{\Omega})} \le C.$$

Note that the $p-$Laplacian can be understood as the invertible map

$$-\Delta_p : W_0^{1,p}(\Omega) \to (W_0^{1,p}(\Omega))^*,$$

which sends $u \in W_0^{1,p}(\Omega)$ to

$$\langle -\Delta_p u, v \rangle = \int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla v dx$$

for all $v \in W_0^{1,p}(\Omega)$. This, together with the continuity of f and g and the fact that $g(s) > 1$ for all $s \in \mathbb{R}$, allows us to define the map

$$C^1(\overline{\Omega}) \ni u \mapsto T_\lambda(u) = (-\Delta_p)^{-1} \left(\frac{\lambda f(u)}{\left(⨍_\Omega g(u) dy\right)^r} \right).$$

We have the following lemma.

Lemma 5.2 *For all $\lambda \ge 0$, the map $T_\lambda : C^1(\overline{\Omega}) \to C^1(\overline{\Omega})$ is completely continuous.*

Proof : Fix $\lambda \ge 0$. For any $u \in C^1(\overline{\Omega})$, since $\frac{\lambda f(u(\cdot))}{\left(⨍_\Omega g(u)dy\right)^r}$ is a bounded function on Ω, we can use Lemma 5.1 to obtain that $T_\lambda u \in C^{1,\alpha}(\overline{\Omega})$ where $\alpha \in (0,1)$. This implies that T_λ is well-defined.

Let $\{u_n\}_{n\in\mathbb{N}}$ be a bounded sequence in $C^1(\overline{\Omega})$. Using the continuity of f and g, we can find $M > 0$, independent of n, such that

$$\left\| \frac{\lambda f(u_n)}{\left(⨍_\Omega g(u_n) dy\right)^r} \right\|_{L^\infty(\Omega)} \le M$$

for all $n \ge 1$. Applying Lemma 5.1 again, we can find $\alpha_1 \in (0,1)$ and $C_1 = C_1(M) > 0$, independent of n, such that

$$\|T_\lambda(u_n)\|_{C^{1,\alpha_1}(\overline{\Omega})} \le C_1$$

for all $n \ge 1$. Using the Ascoli-Azelà theorem for $\{\frac{\partial}{\partial x_i} T_\lambda u_n\}_{n\in\mathbb{N}}$, $i = 1, \cdots, N$, helps us find a convergent subsequence of $\{T_\lambda u_n\}_{n\in\mathbb{N}}$ in $C^1(\overline{\Omega})$. The compactness of T_λ follows.

Finally, if we assume that the sequence $\{u_n\}_{n\in\mathbb{N}}$ in the paragraph above converges to u in $C^1(\overline{\Omega})$ then its is obvious that

$$\int_\Omega |\nabla u_n|^{p-2}\nabla u_n \cdot \nabla v dx \to \int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v dx,$$

$$\int_\Omega \lambda f(u_n) v dx \to \int_\Omega f(u) v dx,$$

for all $v \in E$, and

$$\int_\Omega g(u_n) dy \to \int_\Omega g(u_n) dy$$

as n tends to ∞. We obtain the continuity of T_λ.

Noting that the equation

$$u = T_0(u)$$

has only one solution, which is identically 0, we can employ the arguments based on the Leray Schauder continuation theorem (see [49]) to obtain the following theorems.

Theorem 5.3 *Problem (5.1) has an unbounded continuum of solutions in* $[0,\infty) \times C^1(\overline{\Omega})$.

Theorem 5.4 *Problem (5.2) has an unbounded continuum of solutions in* $[0,\infty) \times C^1(\overline{\Omega})$.

5.2 Using fixed point theorems and the cutting off technique

The aim of this section is to find $\Lambda > 0$ so that (5.2) is solvable for all $\lambda \in [0, \Lambda]$. In order to do that, we proceed as follows. We first find a solution of

$$\begin{cases} -\Delta_p \overline{u} &= \Lambda e^{\overline{u}} \quad \text{in } B, \\ \overline{u} &= 0 \quad \text{on } \partial B. \end{cases} \tag{5.3}$$

where B is a ball containing Ω. Noting that $\overline{u} \geq 0$ and hence

$$\Lambda e^{\overline{u}} \geq \frac{\lambda e^{\overline{u}}}{\left(⨍_\Omega e^{\overline{u}} dy\right)^r}$$

for all $\lambda \in [0, \Lambda]$, we may consider $\overline{u}$ as a supersolution of (5.2). Then, we employ the cutting off technique and fixed point theorems to seek the desired solution. All details are provided here.

Let $B \subset \mathbb{R}^N$ be a ball containing Ω. Applying the study of the Liouville-Bratu-Gelfand problem in [29], we can find $\Lambda > 0$ and $\overline{u} \in C^1(\Omega) \cap C(\overline{\Omega})$ solving (5.3). Noting that $e^{\overline{u}(\cdot)}$ is a nonnegative function and applying the weak comparison principle, we have $\overline{u} \geq 0$ in B. Thus, its restriction on $\partial\Omega$ is nonnegative.

Define

$$\tilde{f}(x,s) = \begin{cases} e^{\overline{u}(x)} & s > \overline{u}(x), \\ e^s & 0 \le s \le \overline{u}(x), \\ 1 & s < 0, \end{cases}$$

for a.e. $x \in \Omega$, all $s \in \mathbb{R}$ and $\tilde{T}_\lambda : W_0^{1,p}(\Omega) \to W_0^{1,p}(\Omega)$ as

$$\tilde{T}_\lambda u = (-\Delta_p)^{-1} \left(\frac{\lambda \tilde{f}(x,u)}{\left(\fint_\Omega \tilde{f}(y,u)dy \right)^r} \right)$$

for all $u \in W_0^{1,p}(\Omega)$ and $\lambda \in [0,\Lambda]$.

Lemma 5.5 *For all $\lambda \in [0,\Lambda]$, $\tilde{T}_\lambda : W_0^{1,p}(\Omega) \to W_0^{1,p}(\Omega)$ has a fixed point.*

Proof : Using the continuity of $\overline{u}$ on $C(\overline{\Omega})$, we can find $M > 0$ such that

$$\sup\{\tilde{f}(x,u(x)), \tilde{f}(x,u(x)) : u \in L^p(\Omega), x \in \Omega\} \le M.$$

This and Lebesgue's dominated convergence theorem help us to see that the Nemytskii operator

$$N_{\tilde{f}} : L^p(\Omega) \to L^{\frac{p}{p-1}}(\Omega)$$

with

$$N_{\tilde{f}} u = \frac{\lambda \tilde{f}(x,u)}{\left(\fint_\Omega \tilde{f}(y,u)dy \right)^r}$$

is continuous. It follows that $\tilde{T}_\lambda$ is completely continuous because it can be decomposed as

$$\begin{array}{ccccccccc} \tilde{T}_\lambda : & W_0^{1,p}(\Omega) & \hookrightarrow & L^p(\Omega) & \to & L^{\frac{p}{p-1}}(\Omega) & \to & (W_0^{1,p}(\Omega))^* & \to & W_0^{1,p}(\Omega) \\ & u & \mapsto & u & \mapsto & N_{\tilde{f}}u & \mapsto & N_{\tilde{f}}u & \mapsto & (-\Delta_p^{-1})(N_{\tilde{f}}u). \end{array}$$

We now use the uniform boundedness of $\tilde{f}$ on $\Omega \times \mathbb{R}$ again to conclude that the set

$$\{\|\tilde{T}_\lambda u\| : u \in W_0^{1,p}(\Omega)\}$$

is bounded in $\mathbb{R}$. Define

$$R = \sup\{\|\tilde{T}_\lambda u\| : u \in W_0^{1,p}(\Omega)\},$$

and

$$B_{R+1} = \{u \in W_0^{1,p}(\Omega) : \|u\| < R+1\}.$$

Then, for all $u \in \partial B_{R+1}$, $\|u\| > \|\tilde{T}_\lambda u\|$. Thus,

$$\deg(\mathrm{Id} - \tilde{T}_\lambda, B_{R+1}, 0) = \deg(\mathrm{Id}, B_{R+1}, 0) = 1.$$

It follows that $\tilde{T}_\lambda$ has a fixed point for all $\lambda \in [0,\Lambda]$.

Let u_λ be the fixed point of $\tilde{T}_\lambda$ obtained in Lemma 5.5, $\lambda \in [0,\Lambda]$. We have the lemma.

Lemma 5.6 *$0 \le u_\lambda \le \overline{u}$ in Ω for all $\lambda \in [0, \Lambda]$.*

Proof : The fact that u_λ is nonegative is deduced from the weak comparison principle and the following equality and inequality

$$-\Delta_p u_\lambda = \frac{\lambda \tilde{f}(x, u_\lambda)}{\left(⨍_\Omega \tilde{f}(y, u_\lambda) dy \right)^r} \ge 0.$$

On the other hand, we may use $(u_\lambda - \overline{u})^+ \in W_0^{1,p}(\Omega)$ as a test function in the equation

$$u_\lambda = \tilde{T}_\lambda u_\lambda$$

to obtain

$$\begin{aligned}
\int_\Omega |\nabla u_\lambda|^{p-2} \nabla u_\lambda \cdot \nabla (u_\lambda - \overline{u})^+ dx &= \int_\Omega \frac{\lambda \tilde{f}(x, u_\lambda)(u_\lambda - \overline{u})^+}{\left(⨍_\Omega \tilde{f}(y, u_\lambda) dy \right)^r} \\
&\le \int_\Omega \Lambda f(\overline{u})(u_\lambda - \overline{u})^+ dx \\
&= \int_\Omega |\nabla \overline{u}|^{p-2} \nabla \overline{u} \cdot (u_\lambda - \overline{u})^+ dx.
\end{aligned}$$

Here, the first inequality is implied by $e^s > 1$ for all $s \ge 0$ and the last equality is deduced by the definition of $\overline{u}$. It follows that

$$\int_{\Omega^+} [|\nabla u_\lambda|^{p-2} \nabla u_\lambda - |\nabla \overline{u}|^{p-2} \nabla \overline{u}] \cdot \nabla (u_\lambda - \overline{u}) dx \le 0,$$

where

$$\Omega^+ = \{x \in \Omega : u_\lambda(x) > \overline{u}(x)\}.$$

Since the integrand is nonnegative, the integral above is nonpositive if either its integrand is 0 or $|\Omega^+| = 0$. Both cases show $u_\lambda \le \overline{u}$ a.e. in Ω.

We are now in the position to state the theorem.

Theorem 5.7 *For all $\lambda \in [0, \Lambda]$, problem (5.2) has a solution.*

Proof : The theorem follows by the Lemma 5.5 and Lemma 5.6 and the definition of $\tilde{f}$.

The question how far λ can be away from 0 such that the existence result for (5.2) is still guaranteed is answered only in some special cases. The following proposition shows that λ may be either bounded or unbounded.

Proposition 5.8 *([46]) If Ω is star-shaped with respect to the origin with $N \ge 3$ and $r \le 1$, then there exists $\overline{\lambda} > 0$ such that*

$$\begin{cases} -\Delta u = \frac{\lambda e^u}{\left(⨍_\Omega e^u dy \right)^r} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{5.4}$$

has no solution for $\lambda > \overline{\lambda}$.

On the other hand, if Ω *is an annulus domain and* $r \geq 1$*, solution of (5.4) exists for all* $\lambda \geq 0$.

CHAPTER 6

CONCLUSION

In this dissertation, we have established some new results. Besides presented here, many of them may be found in [43, 44, 45].

The first contribution of this dissertation is the extension of the results in [13, 27], indicating necessary and sufficient conditions that guarantee the existence of multiple bounded solutions to boundary value problems of the form

$$\begin{cases} -\Delta u &= f(u) \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{cases}$$

where f is a continuous function. We have proved that these results are still true when the Laplacian is replaced by the $p-$ Laplacian and have applied them to study singular semipositone problems.

During the past several years, several studies concerning singular elliptic boundary value problems have appeared, e.g., [1, 11, 12, 21, 24, 25, 33, 34, 52, 53, 56, 58]. All of these contributions have been very ad hoc in nature with methods specifically designed for the the nonlinearities at hand. Our plan was to design a general framework and theory which would encompass the contributions mentioned. This we have achieved by extending the classical method of sub-supersolutions of [36, 37, 38, 43] so it may be used to solve singular elliptic problems. We highlight that our theorems are applicable to problems which also involve convection terms subject to growth conditions motivated and patterned after Bernstein or Nagumo growth conditions. The existence of extremal solutions has been discussed, as well. We emphasize once more that our general results may be used to deduce the results obtained earlier in [1, 11, 12, 21, 24, 25, 33, 34, 52, 53, 56, 58].

In the final chapter we have shown how these methods may be employed to study nonlocal boundary value problems motivated by the Liouville-Bratu-Gelfand problem and the problem modelling shear bandings. For such problems we have shown the existence of an unbounded continuum of solutions and given estimates on the range of values of the parameter for which solutions exist.

APPENDIX

A.1 Classical sub-supersolution theorems for Dirichlet boundary problems

As mentioned in Chapter 3, sub-supersolution theorems in [36, 37, 38] play important roles to our main results. Although such sub-supersultion theorems can be applied to problems with Dirichlet, Neumann and Robin conditions (as sumerized in [50]), we only recall here the one, applicable to Dirichlet boundary problems, for simplicity. Let the Carathéodory mapping $A : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ satisfy Leray-Lions conditions as in Section 3.1 and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function. We first have the definitions of sub-supersolutions and solutions, in the weak sense, to

$$\begin{cases} -\mathrm{div} A(x, \nabla u) & = & f(x,u) & \text{in } \Omega, \\ u & = & 0 & \text{on } \partial\Omega. \end{cases} \tag{A.1}$$

Definition A.1 *A function $u \in W^{1,p}(\Omega)$ is called a subsolution (supersolution) of (A.1) if:*

(i) $u|_{\partial\Omega} \leq (\geq) 0$,

(ii) $f(\cdot, u) \in (W_0^{1,p}(\Omega))^*$,

(iii) for all nonnegative functions $v \in W_0^{1,p}(\Omega)$,

$$\int_\Omega A(x, \nabla u) \cdot \nabla v dx \leq (\geq) \int_\Omega f(x,u) v dx.$$

Definition A.2 *A function $u \in W_0^{1,p}(\Omega)$ is called a solution of (A.1) if for all functions $v \in W_0^{1,p}(\Omega)$,*

$$\int_\Omega A(x, \nabla u) \cdot \nabla v dx = \int_\Omega f(x,u) v dx.$$

We have the theorem.

Theorem A.3 *Assume that problem (A.1) has k subsolutions $\underline{u}_i$, $i = 1, \cdots, k$, and l supersolutions $\overline{u}_1, \cdots, \overline{u}_l$, $k, l \geq 1$, such that*

$$\underline{u} := \max\{\underline{u}_i : i = 1, \cdots, k\} \leq \overline{u} := \min\{\overline{u}_j : j = 1, \cdots, l\} \qquad \text{in } \Omega.$$

Assume further that there exists a function $a_3 \in L^{q'}(\Omega)$, where q' is given by $\frac{1}{q'} + \frac{1}{q} = 1$, $q \in (1, p^*)$, such that

$$|f(x, s, \xi)| \leq a_3(x)$$

for all $s \in [\underline{u}_0(x), \overline{u}_0(x)]$, *for a.e.* $x \in \Omega$, *where*

$$\begin{aligned} \underline{u}_0 &:= \min\{\underline{u}_1, \cdots, \underline{u}_k\}, \\ \overline{u}_0 &:= \max\{\overline{u}_1, \cdots, \overline{u}_l\}. \end{aligned}$$

Then (A.1) has a minimal solution u_* *and a maximal solution* u^* *with respect to the pair* $\underline{u}$ *and* $\overline{u}$.

A.2 A $W^{1,p}-$ priori bound

We establish here a result concerning a priori bounds of solutions, with respect to $W_0^{1,p}(\Omega)$, for L^∞ bounded solutions of (3.7) under a growth condition imposed on f, suggested by a Bernstein or Nagumo growth condition used frequently in the study of nonlinear ordinary differential equations. This type of growth condition appears to have been first used by Bernstein and then extended by Nagumo (see [14], [26], [51]). The result is important in its own right, since it can be used to deduce Lemma 3.24, showing the compactness of a set of solutions to a class of nonlinear elliptic equations in the appropriate Sobolev space. The a priori bound is established using ideas from [54].

Proposition A.4 *Consider the problem*

$$\begin{cases} -div\ A(x, \nabla u) &= f(x, u, \nabla u) \quad in\ \Omega, \\ \qquad\qquad u &= 0 \qquad\qquad\quad on\ \partial\Omega, \end{cases} \tag{A.2}$$

where A *satisfies the same conditions, as stated in Section 3.1, and* f *is a Carathéodory function which satisfies the following requirement: For some positive number* M, *there exist a function* $a_3 \in L^1(\Omega)$ *and a constant* $b_3 > 0$ *such that*

$$|f(x, s, \xi)| \leq a_3(x) + b_3|\xi|^p,$$

for all $s \in [-M, M]$, $\xi \in \mathbb{R}^N$, *for a.e.* $x \in \Omega$. *Then there exists a constant* C, *depending on* M, a_i, *and* b_i, $i = 1, 2, 3$, *such that: If* u *is any weak solution* $u \in W_0^{1,p}(\Omega)$ *of (A.2) with* $|u(x)| \leq M$ *for a.e.* $x \in \Omega$, *it follows that*

$$\|u\| \leq C.$$

Proof : Let $M > 0$ be given and let u be such a weak solution. Choose $v_t = e^{tu^2}u$, $t > 0$, as a test function for (A.2) and obtain

$$\begin{aligned}\int_\Omega e^{tu^2}(1+2tu^2)A(x,\nabla u)\cdot\nabla u dx &\le \int_\Omega e^{tu^2}(a_3+b_3|\nabla u|^p)|u|dx\\ &\le C_1+b_3\int_\Omega e^{tu^2}|\nabla u|^p|u|dx,\end{aligned}$$

where

$$C_1 = Me^{tM^2}\int_\Omega a_3 dx.$$

This and condition (3.6) show that

$$\int_\Omega e^{tu^2}(1+2tu^2)(b_2|\nabla u|^p - a_2)dx \le C_1 + b_3\int_\Omega e^{tu^2}|\nabla u|^p|u|dx.$$

Thus,

$$b_2\int_\Omega e^{tu^2}(1+2tu^2)|\nabla u|^p dx \le C_2 + b_3\int_\Omega e^{tu^2}|\nabla u|^p\left(\frac{\epsilon}{2}+\frac{u^2}{2\epsilon}\right)dx,$$

for any $\epsilon > 0$, where

$$C_2 = C_1 + e^{tM^2}(1+2tM^2)\int_\Omega |a_2|dx.$$

We now choose $\epsilon = \frac{b_3}{4tb_2}$ and t large so that

$$b_2 - \frac{\epsilon b_3}{2} > 0$$

and

$$\int_\Omega e^{tu^2}\left(b_2 - \frac{\epsilon b_3}{2}\right)|\nabla u|^p dx \le C_2.$$

The proposition follows by noting that $e^{tu^2} \ge 1$ for a.e $x\in\Omega$.

Remark A.5 The growth condition imposed on f here is more general than condition (3.8) since a_3 is allowed to belong to $L^1(\Omega)$.

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